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A Note on Calculation of CVaR for Student's Distribution

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Abstract

This study provides an analytical formula for CVaR, calculated for t-type distributions with non-integer degrees of freedom. We generalize standard formulas, calculated in assumption of normal log-returns (see, e.g. Jorion, 2000) without compromising on difficulty of the calculation procedure involved. We also extend results of Heikkinen and Kanto (2002) to show the impact of kurtosis on values of CVaR. Results are summarized in a closed-form formula which can be effortlessly used by risk managers in evaluation risk exposures for a family of heavy tailed distributions. Examples of calculations are included.

1 Introduction

Risk managers and regulators need measures for risk. There are two common measures of risk: Value at Risk (VaR) and Conditional Value at Risk ($CVaR$) (Jorion, 2000; Artzner et al., 1999). The former is the lower bound that is reached with given probability, usually 95%, 97,5%, 99% or 99,5%. The latter gives the expected loss assuming that the lower bound is reached. The traditional way is to assume returns to be Normal. However, in practice this assumption seldom holds, because the tails are heavier than in the Normal case. One possible alternative would be Student's t-distribution (Student, 1908), which has light heavy tails. Student, in his seminal article

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considered distributions with integer degrees of freedom, but mathematically this is not necessary. In this paper, we allow degrees of freedom be non-integer. A nice property of this class of distributions is that kurtosis and degrees of freedom have a simple relationship. Therefore, degrees of freedom can easily be estimated using the moment method. In practice, the kurtosis is often larger than six leading to non-integer degrees of freedom between four and five.

The critical values of Student's t-distribution with integer values are well reported in standard textbooks. Recently, Heikkinen and Kanto (2002) reported them with several non-integer values. In this paper a compact formula for CVaR with non-integer degrees of freedom is presented. It shows that if the data is heavy tailed, i.e. has large kurtosis, the CVaR calculated using the Normal assumption may differ significantly from the t-distributed one, which takes high kurtosis in account.

Section 2 provides proofs and formulas. In Section 3 we examplify the findings. Section 4 concludes.

2 Calculation of VaR and CVaR for Student's t-distributions

VaR literature (see e.g. Alexander, 1998; Jorion, 2000) often assumes logarithmic returns to be normally distributed, implying excess kurtosis to be zero. This condition is too restricitve and does not get empirical support. Standard statistical tests suggest heavy tails for most of financial time series. Student's t-distributions constitute a family that allows for modelling of non-zero excess returns. Density (see Abramowitxand Stegun, 1971) of non-central Student t-distribution has the following form

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\beta\nu}}(1 + \frac{(x-\mu)^2}{\beta\nu})^{-(1+\nu)/2}, \quad (1)$$

where μ is a location parameter, β is a dispersion parameter, and ν is a shape parameter, or degrees of freedom. Standard t-distribution assumes $\mu = 0$, $\beta = 1$, and ν to be an integer. We follow Heikkinen and Kanto (2002) by assuming non-integer degrees of freedom and applying method of moments for estimation of parameters.

Calculation of CVaR for normal random variables boils down to hazard rate evaluation at significance level quantile, i.e.

$$CVaR_n = -\frac{f(q)}{F(-q)} \tag{2}$$

By construction CVaR should be always larger than VaR. Simple check shows that classical $CVaR_n$ formula for normal random variable produces misleading results if applied directly to t-distributions, i.e. one can easily find quantiles of t-distribution for which this survival intensity is smaller than the estimate value for VaR_t.

We calculate analytically the correction term which fixes the classical formula for t-distributions. CVaR of t-distribution (see 1) as follows

$$CVaR_t = \int_{-\infty}^{q} xf(x)dx = \int_{-\infty}^{q} x\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\beta\nu}}(1+\frac{x^2}{\beta\nu})^{-\frac{1+\nu}{2}}dx$$

Let $\nu > 1$. Straightforward integration by substitution $y = x^2/\beta\nu$ results in

$$\int_{-\infty}^{q} xf(x)dx = -\frac{\beta\nu}{\nu-1}\left(1+\frac{q^2}{\beta\nu}\right)f(q)$$

Furthermore, since $\int_{-\infty}^{q} f(x)dx = F(-q)$ by symmetry of t-distribution, we have

$$CVaR_t = -\frac{\beta\nu}{\nu-1}\left(1+\frac{q^2}{\beta\nu}\right)\frac{f(q)}{F(-q)}$$

The second moment of the t-distribution can be estimated as $m_2 = \frac{\beta\nu}{\nu-2}$. Assuming $\nu > 2$, we get the moment estimator $\beta = \frac{\nu-2}{\nu}m_2$, yielding

$$CVaR_t = -\left((1-\omega)\,m_2 + \omega q^2\right)\frac{f(q)}{F(-q)}$$

where $\omega = \frac{1}{\nu-1}$.

Let $\nu > 4$. Simple calculation yields $\omega = kur/(6{+}3kur)$, as $\nu = 4{+}6/kur$. Using moment estimators s^2, $\widehat{kur}$ and $\widehat{\omega} = \widehat{kur}/(6+3\widehat{kur})$, we finally get the estimator

$$\widehat{CVaR}_t = -\left((1-\widehat{\omega})s^2 + \widehat{\omega}q^2\right)\frac{f(q)}{F(-q)}, \tag{3}$$

that can be thought of as a weighted sum of sampled variance and squared quantile.

Figure 1 The weights as a function of kurtosis in formula (3)

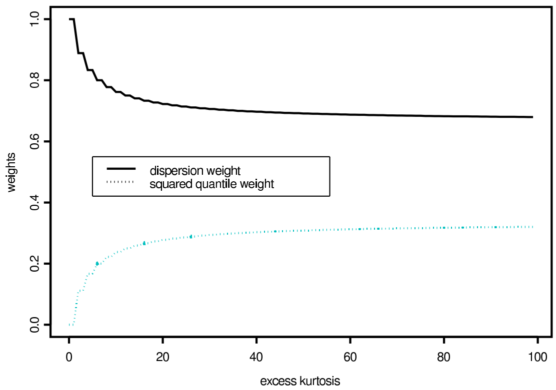


Figure 1 demonstrates that the impact of the "quantile" term is growing with growth of excess kurtosis but this growth is bounded by weight of 1/3. One can see that kurtosis value of 40 is already big enough to use formula 3 in the limiting form

$$\widehat{CVaR}_t = -(\frac{2}{3}s^2 + \frac{1}{3}q^2)\frac{f(q)}{F(-q)} \tag{4}$$

Another limiting case arises when excess kurtosis iz zero. Formula (3) can be rewritten as a function of kurtosis

$$\widehat{CVaR}_t = -(\frac{2\widehat{kur}+6}{6+3\widehat{kur}}s^2 + \frac{\widehat{kur}}{6+3\widehat{kur}}q^2)\frac{f(q)}{F(-q)}, \tag{5}$$

and $\widehat{kur} = 0$ suggests no impact of "quantile" term. The resulting formula is a classical CVaR formula 2 for normal log-returns, i.e. hazard evaluated at the appropriate quantile.

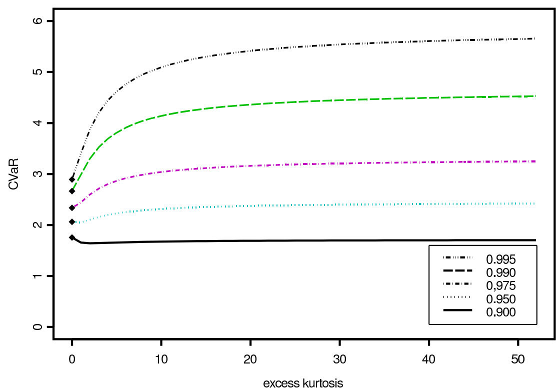
Formulae 3 and 4 are non-biased estimates. They play the same role for calculation of conditional value at risk as $VaR_t = t_\alpha^\nu\sqrt{\hat{\beta}}$, where $\hat{\beta} = (\frac{3+\hat{kur}}{3+2\hat{kur}})s^2$ (see Heikkinen and Kanto, 2002) for calculation of value at risk.

3 Simulation Results

Results of numerical integration for non-integer degrees of freedom and correcting coefficient values for calculation of VaR_t have been summarized by Table 2 in Heikkinen and Kanto (2002) . The effect of kurtosis on VaR_t at different probability levels has been presented in Figure 1 of the same paper.

Our first objective is to report effect of excess kurtosis on $CVaR_t$ at different probability levels and compare these findings with results one would get in assumption of normal log-returns. Formula 3 is the key for producing Figure 2.

Figure 2 The effect of excess kurtosis on CVaR_t, s=1



Heikkinen and Kanto (2002) have found that effect of kurtosis on value of VaR_t is almost insignificant for level 97.5%, while it has a somewhat decreasing effect at lower levels, and increasing effect for higher levels. In contrast to their findings, higher kurtosis increases $CVaR_t$ at all levels, with a mild exception for 0.9 significance level, when $CVaR_t$ slightly decreases for small values of kurtosis in order to recover its level for higher ones. This increasing behaviour $CVaR_t$ manifests itself stronger as significance level approaches 1. For levels above 97.5%, increasing effect becomes very strong in comparison to VaR_t calculations.

Figure 2 provides a clear way to see the differences between $CVaR_t$ and $CVaR_n$. Since the excess kurtosis is zero for normal random variables, the corresponding values of $CVaR_n$ are points on the graph. They are obviously ordered in the way that the higher point corresponds to higher level. By definition of Student's t-distribution, when excess kurtosis tends to zero, the whole distribution tends to normal distribution. Surprisingly, one can see that $CVaR_t < CVaR_n$ for 90% level. This effect is mild but present. As one increases the level, graphs match intuition better: effect of excess kurtosis increases the risk value.

Figure 3: The effect of kurtosis on ratio of (C)VaR_t to (C)VaR_n

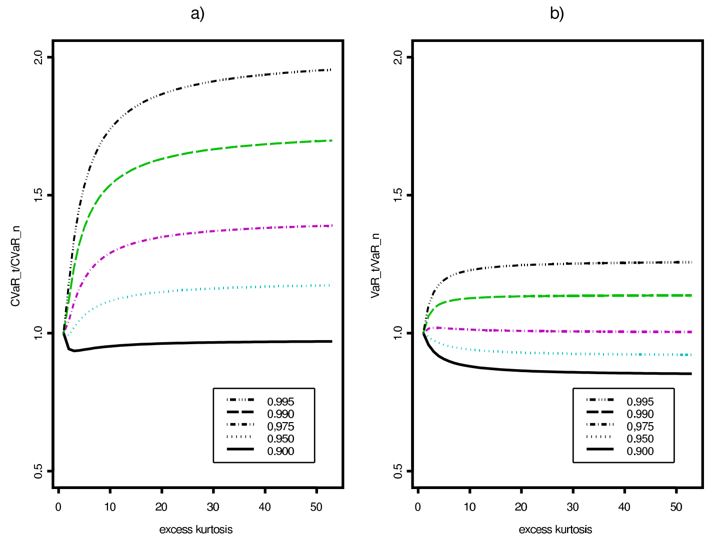


Figure 3 scales effect of the excess kurtosis on $\frac{(C)VaR_t}{(C)VaR_n}$. Intuitive hypothesis that higher kurtosis implies higher $(C)VaR_t$ surprisingly fails for both ratios with more pronounced effect observed for $\frac{VaR_t}{VaR_n}$.

All lines of Figures 3a, 3b are ordered as functions of p-values: larger p-value corresponds to larger value for $\frac{(C)VaR_t}{(C)VaR_n}$. This observation makes analysis simple. Another important observation is that for all p-values $\frac{CVaR_t}{CVaR_n} > \frac{VaR_t}{VaR_n}$.

The most interesting question to answer is when the ratio equals to one, i.e. when $(C)VaR_t = (C)VaR_n$. Answers are different for CVaR and VaR. In line with Heikkinen and Kanto (2002), Figure 3b suggests $p = 0.975$ to be the level at which $VaR_t \approx VaR_n$ for all values of kurtosis. Alternatively, $VaR_t > VaR_n$ for $p > 0.975$, while $VaR_t < VaR_n$ for $p < 0.975$.

Somewhat surprisingly, similar effect is observed for the ratio $\frac{CVaR_t}{CVaR_n}$ (see Figure 3a). It is not that straightforward as for $\frac{VaR_t}{VaR_n}$: the threshold level depends upon value of kurtosis and belongs to p-value interval $[0.9, 0.95]$. $\frac{CVaR_t}{CVaR_n}$ takes values on both sides of 1 for the same p-level. For instance, at 0.9 level, $\frac{CVaR_t}{CVaR_n}$<1 for all levels but if $p = 0.95$, $\frac{CVaR_t}{CVaR_n} > 1$ for $kur > 1$, while $\frac{CVaR_t}{CVaR_n} = 0.99$ for $kur = 1$.

Finally, we also demonstrate our findings using the same Nokia stock which has been used to illustrate effect of excess kurtosis on calculation of VaR (see Heikkinen and Kanto (2002)). The stock has been followed for four and a half year with history of large fluctuations on monthly scale. We apply formula 5 with $s = 20\%$ and $kur = 10$, indicating a Student's t-distribution with 4.6 degrees of freedom.

Table 1: Monthly estimates for Nokia stock

Confidence level	90%	95%	97.5%	99%	99.5%
CVaR (Normal)	38.6%	45.4%	51.4%	58.6%	63.6%
CVaR(Student, kur=10)	36.8%	50.7%	66.4%	90.1%	110.7%

Table 1 summarizes the results. It indicates rapidly growing difference in value of $CVaR$ for higher p-values, contrasting with much smoother behaviour for the VaR estimates.

4 Concluding Remarks

This article presents a simple closed form formula for calculation of conditional value at risk (CVaR) for Student's t-distribution. Formula 3 is a weighted average of the estimated variance and the square of the allowed critical point. It contains a classical formula for calculation of CVaR for normal distributions as a partial case. Since financial data usually has a feature of heavy tails, it is of interest for practicians. Assuming finite kurtosis, the weights are easy to estimate from the data and therefore formula 3 provides a quick and useful tool in risk management.

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