Andriy Andreev – Antti Kanto – Pekka Malo

ON CLOSED-FORM CALCULATION OF CVAR

Andriy Andreev* – Antti Kanto** – Pekka Malo**

ON CLOSED-FORM CALCULATION OF CVAR

* Swedish School of Economics

** Helsinki School of Economics,
Quantitative Methods in Economics and
Management Science

September
2005

ON CLOSED-FORM CALCULATION OF CVAR

ANDRIY ANDREEV [1]

Swedish School of Economics, P.O. Box 479, FIN-00101

e-mail: andriy.andreev@hanken.fi

ANTTI KANTO

Helsinki School of Economics, P.O. Box 1210, FIN-00101

e-mail: antti.kanto@hkkk.fi

PEKKA MALO [2]

Helsinki School of Economics, P.O. Box 1210, FIN-00101

e-mail: pekka.malo@hkkk.fi

ABSTRACT. Although Conditional Value-at-Risk has significant advantages over traditional risk measures such as Value-at-Risk, it has not been adopted by practitioners as quickly as expected. One of the reasons slowing down its progress has been the lack of simple tools for its computation. In this paper we consider calculating CVaR when the underlying asset is modelled using a diffusion process with a linear drift and prespecified marginal density. The results are summarized in two closed-form formulas which can be effortlessly applied by risk managers to calculate CVaR for a number of commonly used probability distributions. Example of calculations is included.

KEY WORDS: conditional value-at-risk, coherence, risk measure, expected shortfall

1. INTRODUCTION

The problem of finding risk measures that appropriately penalize the tails has received considerable attention in the last few years. Although Value-at-Risk (VaR) is still widely used for measuring extreme events and integrating disparate sources of risk, its limitations are increasingly recognized (Szegö (2002), Daníelsson (2002)). The main caveat of VaR is that it is not a convex functional when non-elliptical distributions are considered, which makes it inappropriate for portfolio-optimization problems. Further the lack of sub-additivity implies that portfolio diversification may lead to an increase in risk and prevent to add up the VaR of different risk sources. Thus VaR is not coherent in Artzner et al. (1999) sense and regulatory agencies should be careful about insisting its use. As discussed by Rockafellar and Uryasev (2002) a serious shortcoming of VaR is that it merely provides a lowest bound for losses without being able to distinguish between their degrees. Therefore VaR has a bias toward optimism instead of conservatism that ought to prevail in risk management.

In response an alternative risk measure, Conditional Value-at-Risk (CVaR), has been proposed to replace VaR. The definition of CVaR is relatively intuitive: for general distributions CVaR is defined as the weighted average of VaR and the

---

expected losses that are strictly greater than VaR, which in the special case of continuous random variables equals the expected loss given that the loss is greater than or equal to the VaR (Szegö (2002)). Thus CVaR provides an upper bound for VaR. But unlike VaR, CVaR is a coherent risk measure and suitable for stochastic optimization and convex analysis (Rockafellar and Uryasev (2002)). Further Rockafellar and Uryasev (2000) have shown that CVaR can be minimized using linear programming techniques, which makes CVaR an appealing tool for fund managers especially as it is able to account for asymmetric return distributions.

However, despite CVaR's significant advantages over VaR machine, its adoption by practitioners of risk management has not progressed as expected. One of the reasons could be the lack of simple formulas to evaluate CVaR. To alleviate this dilemma, we provide analytical solutions for a large number of commonly used continuous probability distributions. In particular we consider the problem of computing CVaR when the underlying asset has a diffusion process with a linear drift and hence an exponentially decreasing autocorrelation function. While doing so we will draw on the findings of Bibby et al. (2005) and Aït-Sahalia (1996), who have studied the construction and estimation of diffusion processes with exponential autocorrelation functions.

The structure of this paper is as follows. In Section 2 we discuss the definition of CVaR as a coherent risk measure in Artzner et al. (1999) sense. In Section 3 we present proofs and formulas. In Section 4 we provide examples of our findings. Section 5 concludes the paper.

2. CVaR as a coherent risk measure

In their seminal paper Artzner et al. (1999) outline the desirable properties that an ideal coherent measure should have: sub-additivity, translation invariance, positive homogeneity and monotonicity.

Definition 2.1 (Coherence). Functional $\rho : X \longrightarrow \mathbb{R}$ is a coherent risk measure if it has the following properties:

   (1) Subadditivity: $\rho(x + y) \leq \rho(x) + \rho(y)$ for all x,y $\in X$.
   (2) Positive homogeneity: $\rho(\lambda x) = \lambda x$ for all $x \in X$ and $\lambda > 0$, $\lambda \in \mathbb{R}$.
   (3) Monotonicity: if $x \leq y$ then $\rho(x) \leq \rho(y)$ for all x,y $\in X$.
   (4) Transitional invariance: $\rho(x + \alpha r_0) = \rho(x) - \alpha$ for all $x \in X$, $\alpha \in \mathbb{R}$, and all risk-free interest rates r_0.

If ρ satisfies the first two conditions, it is convex. If ρ is convex, transitionally invariant and monotonous it can be called weakly coherent. Any risk measure violating these conditions could lead to serious inconsistencies. To understand their importance, we can interpret them in the light of traditional portfolio theory.

The first condition, subadditivity, is equivalent to portfolio diversification. If it failed, we would be better off by splitting our portfolio in order to decrease risks. As noted by Szegö (2002), measuring risk without subadditivity is like measuring the distance between two points using a rubber band instead of a ruler. The second property, positive homogeneity, is related to liquidity considerations: a large investment λx could be less liquid than the total λx of λ smaller investments in x, which implies $\rho(\lambda x) \geq \lambda \rho(x)$ (Artzner et al. (1999)). This combined with subadditivity leads to equality in the second condition. Monotonicity, then again,

rules out all semi-variance based risk measures. Finally, transitional invariance implies that adding a riskless return of αr_0 to our portfolio reduces risk by α.

Whereas VaR is not even weakly coherent (Szegö (2002), Daníelsson et al. (2001)), CVaR has all properties of definition (2.1). However, we must be careful when defining CVaR, since all of the different definitions are not consistent. The definition depends on whether continuous or general distributions are considered. As pointed out by Rockafellar and Uryasev (2002), the case of general distributions with possible discontinuities requires a more subtle definition. This is an important point to keep in mind, but given that our paper is restricted to continuous marginal densities it is sufficient to define CVaR as the expected loss given that the loss is greater or equal to the VaR. Traditionally CVaR is calculated only for the negative tail. However, as in the case of interest rate risks we might be more interested in calculating the risks of the positive tail. Therefore we extend the definition of CVaR as follows.

Definition 2.2 (CVaR). For a continuous random variable X, CVaR_q is given by the expected value

$$(2.1) \qquad CVaR_q = \mathbf{E}[X|A],$$

where A denotes sets $X \leq q$ or $X \geq q$ for negative and positive tails, respectively.

3. The main result

In this section we derive exact formulas for CVaR given a wide set of prespecified marginal distributions. The derivation of the analytical formulas builds on the findings by Bibby et al. (2005) and Aït-Sahalia (1996), who have considered construction and estimation of diffusions with given exponential autocorrelations and marginal densities.

Following Bibby et al. (2005) let us construct the diffusion process X such that the marginal distribution is concentrated on the set $(l, u) \subset \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, and has a prespecified density f with respect to the Lebesgue measure on the support (l, u) satisfying the following condition. We further require that the density is continuous, bounded, and strictly positive on the support, zero outside, and has a finite variance.

Now consider X_t as a solution to a mean reverting process

$$(3.1) \qquad dX_t = -\theta(X_t - \mu)dt + \sqrt{v(X_t)}dW_t, \qquad t \geq 0, \theta > 0$$

where $\mu \in (l, u)$ and v is a non-negative function defined on the set (l, u) such that following conditions hold

Condition 3.1.

$$(3.2) \qquad \int_l^u v(x)f(x)dx < \infty$$

where

$$(3.3) \qquad v(x) = \frac{2\theta \int_l^x (\mu - y)f(y)dy}{f(x)} = \frac{2\theta\mu F(x) - 2\theta \int_l^x yf(y)dy}{f(x)},$$

$l < x < u$ and F is the distribution function associated with the density f.

Then the following theorem guarantees that the time series X is stationary and ergodic with invariant density f and autocorrelation function $\exp\{-\theta t\}$.

Theorem 3.2 (Bibby-Skovgaard-Sørensen)**.** *Suppose the probability density f has expectation μ and satisfies Condition 3.1. Then the following holds.*

(1) *The stochastic differential equation given by (3.1) and (3.3) has a unique Markovian weak solution. The diffusion coefficient is strictly positive for all $l < x < u$.*

(2) *The diffusion process X that solves (3.1) and (3.3) is ergodic with invariant density f.*

(3) *Equation (3.2) is satisfied. If $X_0 \sim f$, then X is stationary, $\mathbf{E}[X_{s+t}|X_s = x] = x \exp\{-\theta t\}$, and the autocorrelation function for X is given by*

$$Corr(X_{s+t}, X_s) = \exp\{-\theta t\}, \qquad s, t \geq 0.$$

(4) *If $-\infty < l$ or $u < \infty$, then the diffusion given by (3.1) and (3.3) is the only ergodic diffusion with drift $-\theta(x - \mu)$ and invariant density f. If the state space is $\mathbb{R}$, it is the only ergodic diffusion with drift $-\theta(x - \mu)$ and invariant density f for which (3.2) is satisfied.*

Under the postulates of ergodicity and density invariance the long-term averages of the original process are equal to the corresponding state-space averages and the density of X_t does not depend on t. A necessary condition for a stationary process to be ergodic for the mean is that its autocovariance function decays sufficiently quickly.

By accepting the above conditions we get access to the following analytical formulas which can be used to calculate CVaR for negative and positive tails in a straightforward manner.

Proposition 3.3. *If a time series X is generated by (3.1) such that (3.2) and (3.3) hold, then $CVaR_q$ is obtained from*

$$(3.4) \qquad CVaR_{q,d} = \mu - \frac{v(q)}{2\theta} \frac{f(q)}{F(q)}$$

for the negative tail, $X \leq q$. Similarly for the positive tail, $X \geq q$, we have

$$(3.5) \qquad CVaR_{q,u} = \mu + \frac{v(q)}{2\theta} \frac{f(q)}{1 - F(q)}.$$

Proof. Consider the case $X \geq q$. By construction

$$CVaR_{q,u} = \frac{1}{P(X \geq q)} \int_{\{X \geq q\}} x f(x) dx.$$

From (3.3) we have

$$\int_{\{X \leq q\}} x f(x) dx = \mu F(q) - \frac{v(q) f(q)}{2\theta}.$$

Then

$$\int_{\{X \geq q\}} x f(x) dx = \mu - \mu F(q) + \frac{v(q) f(q)}{2\theta}$$

and the result follows by direct substitution. The case $X \leq q$ is similar. $\qquad\square$

This result implies that given a prespecified marginal density we can easily compute CVaR, if it has drift and diffusion functions expressible in closed form. The reason why this approach is worthwhile to consider is that the assumptions we had to make to get this far were not overly restrictive. Indeed many commonly used

probability distribution functions satisfy them. A number of probability densities satisfying weak regularity conditions can be obtained as marginal distribution by choosing the diffusion function v suitably even in the case of a linear drift coefficient. This is particularly useful when considering parametrized classes of diffusion coefficients. In order to demonstrate how easily this framework can be applied, we present formulas to calculate CVaR for a number of distributions in Table 1.

4. NUMERICAL EXAMPLE

We illustrate our approach in the context of interest rate modelling. The most famous and computationally simple example is the term structure model introduced by Cox, Ingersoll, and Ross (1985; hereafter CIR), which is based on a mean-reverting short-term interest rate:

$$dr_t = (\theta - \kappa r_t)dt + \sigma\sqrt{r_t}dW_t$$

Although this specification is rejected by many studies, it has been found to model the short rate quite reasonably. The idea of modelling short rates with mean reversion is well justified when assuming the interest rates to follow capital stock adjustments. Furthermore, interest rates are not traded which means that the arbitrage relationships do not impose many restrictions against a mean-reverting model. This makes CIR a convenient example given the linear drift requirement of Proposition (3.3).

When κ, θ, and σ are all strictly positive and $\sigma^2 \leq 2\theta$, the square root process has a unique fundamental solution and its marginal density is Gamma and its transition density is a type I Bessel function distribution or non-central χ^2 (Carrasco et al. (2002)). Thus in order to have an empirically reasonable process, we consider the parameter estimates obtained from Gallant and Tauchen (1998): $dr_t = (0.02491 - 0.00285r_t)dt + 0.0275\sqrt{r_t}dW_t$. With this parametrization we get an invariant marginal Gamma density with $\alpha = 65.8777$ and $\lambda = 7.5372$:

$$f(r) = \frac{\lambda^\alpha}{\Gamma(\alpha)}r^{\alpha-1}e^{-\lambda r} = \frac{7.5372^{65.8777}}{\Gamma(65.8777)}r^{64.8777}e^{-7.5372r}.$$

The obtained distribution is almost Gaussian with expected value at $\mu = \alpha/\lambda = 8.7404$ (see Figure 1). The cumulative density function is given by

$$F(r) = \frac{\Gamma(\lambda r; \alpha)}{\Gamma(\alpha)},$$

where $\Gamma(r; \alpha) = \int_0^r y^{\alpha-1}e^{-y}dy$ is an incomplete gamma function.

Now we can compute CVaR for each quantile level $q > 0$ using the following formula (see Table 1)

$$CVaR_q = \frac{\alpha}{\lambda} + \frac{1}{\lambda}q\frac{f(q)}{1 - F(q)} = 8.7404 + 0.1327\frac{qf(q)}{1 - F(q)}$$

The estimates for CVaR using several confidence levels are furnished in Table 3. Acknowledging the fact that the above parametrization produces a Gamma density, which is considerably close to the normal distribution, it is not surprising to find that the difference between quantiles and CVaR estimates narrows down as the risk level increases. In the case of distributions with heavier tails, such as the Student's t-distribution, the converse is true. These effects along with analytical formulas for Student's t-distribution are discussed by Andreev and Kanto (2005).

TABLE 1. CVaR for the most common distributions. In the table, Ei denotes the exponential integral function

Name and support	Density function $f(x)$	Mean μ	Squared diffusion $\upsilon(x)$	CVaR ($q < 0$)	CVaR ($q > 0$)
Normal $(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}}e^{-1/2x^2}$	0	2θ	$-\frac{f(q)}{F(q)}$	$\frac{f(q)}{1-F(q)}$
Student $(-\infty, \infty)$	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left(1+\frac{1}{\nu}x^2\right)^{-\frac{(\nu+1)}{2}}$ $\nu > 0$	0	$\frac{2\theta}{\nu-1}(\nu+x^2)$	$-\frac{\nu+q^2}{\nu-1}\frac{f(q)}{F(q)}$	$\frac{\nu+q^2}{\nu-1}\frac{f(q)}{1-F(q)}$
Laplace $(-\infty, \infty)$	$\frac{\alpha^2-\beta^2}{2\alpha}e^{\beta x-\alpha|x|}$ $\alpha^2 > \beta^2$	$\frac{2\beta}{\alpha^2-\beta^2}$	$\frac{2\theta}{\alpha^2-\beta^2}(1+\alpha|x|+\beta x)$	$\frac{2\beta}{\alpha^2-\beta^2} - \frac{1+\alpha|q|+\beta q}{\alpha^2-\beta^2}\frac{f(q)}{F(q)}$	$\frac{2\beta}{\alpha^2-\beta^2} + \frac{1+\alpha|q|+\beta q}{\alpha^2-\beta^2}\frac{f(q)}{1-F(q)}$
Logistic $(-\infty, \infty)$	$\frac{e^x}{(1+e^x)^2}$	0	$2\theta[(e^x + e^{-x} + 2)\log(1+e^x) -x(1+e^x)]$	$-[(e^q + e^{-q} + 2)\log(1+e^q) -q(1+e^q)]\frac{f(q)}{F(q)}$	$[(e^q + e^{-q} + 2)\log(1+e^q) -q(1+e^q)]\frac{f(q)}{1-F(q)}$
Extreme value $(-\infty, \infty)$	$e^{-x-e^{-x}}$	γ	$2\theta e^x(\gamma - x + e^{-x} + e^{e^{-x}}\,Ei(-e^{-x}))$	$\gamma - [e^q(\gamma - q + e^{-q}\,Ei(-e^{-q})]\frac{f(q)}{F(q)}$	$\gamma + [e^q(\gamma - q + e^{-q}\,Ei(-e^{-q})]\frac{f(q)}{1-F(q)}$
Pareto $(0, \infty)$	$\alpha(1+x)^{-\alpha-1}$ $\alpha > 1$	$\frac{1}{\alpha-1}$	$2\theta\mu x(1+x)$	-	$\frac{1}{\alpha-1}[1+q(1+q)\frac{f(q)}{1-F(q)}]$
Exponential $(0, \infty)$	$\lambda e^{-\lambda x}$ $\lambda > 0$	$\frac{1}{\lambda}$	$\frac{2\theta}{\lambda}x$	-	$\frac{1}{\lambda}+q$
Gamma $(0, \infty)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$ $\alpha \geq 1, \lambda > 0$	$\frac{\alpha}{\lambda}$	$\frac{2\theta}{\lambda}x$	-	$\frac{1}{\lambda}[\alpha+q\frac{f(q)}{1-F(q)}]$
χ^2 $(0, \infty)$	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\frac{\nu}{2}-1}e^{-\frac{1}{2}x}$ $\nu \geq 2$	ν	$4\theta x$	-	$\nu + 2q\frac{f(q)}{1-F(q)}$

TABLE 2. CVaR for the most common distributions (continued). In the table, Φ denotes the standard normal distribution and $\Gamma(x;\alpha)$ is the incomplete gamma function

| Name and support	Density function $f(x)$	Mean μ	Squared diffusion $v(x)$	CVaR $(q<0)$	CVaR $(q>0)$
Inverse Gamma $(0,\infty)$	$\frac{\delta^\lambda}{\Gamma(\lambda)} x^{-\lambda-1} e^{-\delta/x}$ $\delta > 0, \lambda > 1$	$\frac{\delta}{\lambda-1}$	$\frac{2\theta}{\lambda-1} x^2$	-	$\frac{1}{\lambda-1}\left[\delta + q^2 \frac{f(q)}{1-F(q)}\right]$
Inverse Gaussian $(0,\infty)$	$\sqrt{\frac{\lambda}{2\pi x^3}}\, e^{\frac{\lambda(x-\delta)^2}{2\delta^2 x}}$ $\delta > 0, \lambda > 0$	δ	$\frac{4\theta\delta}{f(x)} e^{\frac{2\lambda}{\delta}} \Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\delta}+1\right)\right)$	-	$\delta + 2\delta e^{\frac{2\lambda}{\delta}} \Phi\left(-\sqrt{\frac{\lambda}{q}}\left(\frac{q}{\delta}+1\right)\right) \frac{1}{1-F(q)}$
F $(0,\infty)$	$\frac{\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}}{B\left(\frac{\alpha}{2},\frac{\beta}{2}\right)} \frac{x^{\frac{\alpha}{2}-1}}{(\beta+\alpha x)^{\frac{\alpha+\beta}{2}}}$ $\alpha \geq 2, \beta > 2$	$\frac{\beta}{\beta-2}$	$\frac{4\theta}{\alpha(\beta-2)} x(\beta+\alpha x)$	-	$\frac{\beta}{\beta-2} + \frac{2q(\beta+\alpha q)}{\alpha(\beta-2)} \frac{f(q)}{1-F(q)}$
log-normal $(0,\infty)$	$\frac{1}{\sqrt{2\pi\sigma^2}\,x} e^{-\frac{1}{2\sigma^2}(\log x-\delta)^2}$	$e^{\delta+2\sigma^2}$	$\frac{2\theta\mu}{f(x)}\left(\Phi\left(\frac{\log x-\delta}{\sigma}\right) - \Phi\left(\frac{\log x-\delta}{\sigma}-\sigma\right)\right)$	-	$e^{\delta+2\sigma^2} + \mu\left[\Phi\left(\frac{\log q-\delta}{\sigma} - \sigma\right) - \Phi\left(\frac{\log q-\delta}{\sigma}\right)\right]\frac{1}{1-F(q)}$
Weibull $(0,\infty)$	$cx^{c-1}e^{-x^c}$ $c > 0$	$\Gamma\left(\frac{1}{c}+1\right)$	$\frac{2\theta}{f(x)}\left(\Gamma\left(\frac{1}{c}+1\right)(1-e^{-x^c}) - \Gamma\left(x^c;\frac{1}{c}+1\right)\right)$	-	$\Gamma\left(\frac{1}{c}+1\right) + \frac{1}{1-F(q)}\left[\Gamma\left(\frac{1}{c}+1\right)(1-e^{-q^c}) - \Gamma\left(q^c;\frac{1}{c}+1\right)\right]$

FIGURE 1. Marginal gamma density ($\alpha = 65.8777$, $\lambda = 7.5372$)
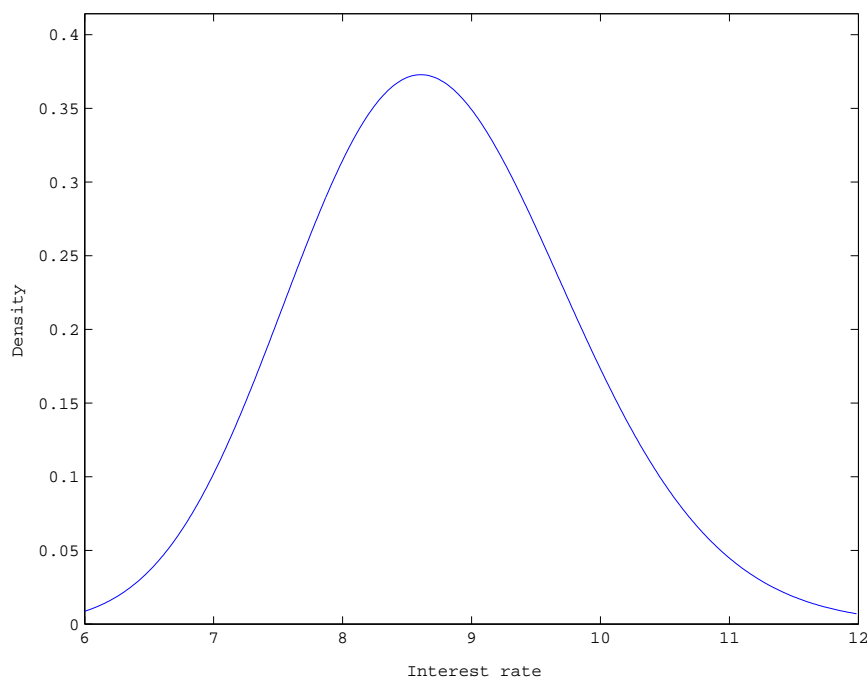
TABLE 3. CVaR estimates for CIR term structure model

Confidence level	90%	95%	97.5%	99%	99.9%
Quantile	10.15%	10.58%	10.97%	11.44%	12.45%
CVaR (Gamma)	10.73%	11.11%	11.46%	11.89%	12.83%

5. CONCLUDING REMARKS

This article presents simple analytical formulas for calculation of conditional value at risk (CVaR) for diffusions with a linear drift and a given marginal density. The results are summarized in Tables 1 and 2 for the most common probability density functions ranging from Student's t-distribution to variety of fat tailed distributions such as Weibull and Laplace distributions. Since financial data usually has a feature of heavy tails, these formulas are of interest for practitioners. So far, we have restricted our study for diffusions with linear drift only, but as an issue for future research it will be of interest to consider more complicated autocorrelation structures than the exponential case.

REFERENCES

Aït-Sahalia, Y. "Nonparametric Pricing of Interest Rate Derivative Securities." *Econometrica* 64 (1996), 527–560.

Andreev, A., and A. Kanto. "Conditional value-at-risk estimation using non-integer values of degrees of freedom in Student's t-distribution." *Journal of Risk* 2 (2005), 55-62.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath. "Coherent Measures of Risk." *Mathematical Finance* 9 (1999), 203–228.

Bibby, B., I. Skovgaard, and M. Sørensen. "Diffusion-type Models with Given Marginal and Autocorrelation Function." *Bernoulli* 11 (2005), 191–220.

Carrasco, M., M. Chernov, J.-P. Florens, and E. Ghysels. "Efficient Estimation of Jump Diffusions and General Dynamic Models with a Continuum of Moment Conditions." Working paper (2002).

Cox, J., J. Ingersoll, and S. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53 (1985), 385–402.

Daníelsson, J. "The Emperor Has No Clothes: Limits to Risk Modelling." *Journal of Banking and Finance* 26 (2002), 1273–1296.

Daníelsson, J., P. Embrechts, C. Goodhart, C. Keating, F. Muennich, O. Renault, and H.-S. Shin. "An Academic Response to Basel II." Special Paper No. 130 (2001), FMG and ESRC, London, available from www.riskresearch.org.

Gallant, A., and G. Tauchen. "Reprojecting Partially Observed Systems with Application to Interest Rate Diffusions." *Journal of American Statistical Association* 93 (1998), 10–24.

Rockafellar, R.T., and S. Uryasev. "Optimization of Conditional Value-at-Risk." *Journal of Risk* 2 (2000), 21–41.

Rockafellar, R.T., and S. Uryasev. "Conditional Value-at-Risk for General Loss Distributions." *Journal of Banking and Finance* 26 (2002), 1443–1471.

Szegö, G. "Measures of Risk." *Journal of Banking and Finance* 26 (2002), 1253–1272.