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APPROXIMATIONS OF THE NONDOMINATED SET

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Interactive Approach Utilizing Approximations of the Nondominated Set

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Abstract

We present a new interactive approach for solving multicriteria optimization problems. We produce rough approximations of the nondominated set and let the decision maker indicate with the help of reference points where to refine the approximation. In this way, (s)he iteratively directs the search towards the best nondominated solution. After the decision maker has identified the most interesting region of the nondominated set, the final solution can be fine-tuned with existing interactive methods. We suggest different ways of updating the reference point as well as discuss visualizations that can be used in comparing different nondominated solutions. The new method is computationally inexpensive and easy to use for the decision maker.

Keywords: *multiple objective programming, multiobjective optimization, approximation, interactive methods, reference point, hybrid*

1 Introduction

In multicriteria optimization, the problem is to find the best compromise solution in the presence of several conflicting criteria, see, for example, Ballestero and Romero (1998), Chankong and Haimes (1983), Cohon (1978), Hwang and Masud (1979), Miettinen (1999), Sawaragi *et al.* (1985), Statnikov and Matusov (1995), Steuer (1986) and Szidarovszky *et al.* (1986). In such problems, there is no single well-defined optimal solution but several mathematically equally good solutions,

so-called nondominated, noninferior or Pareto optimal solutions can be identified. To be able to find the best among them, we need additional preference information from a human decision maker who knows the problem domain.

Methods developed for solving multicriteria optimization problems can be classified according to the role of the decision maker in four classes, see, for example, Hwang and Masud (1979) and Miettinen (1999). In so-called no-preference methods there is no decision maker available and the final solution is some neutral compromise. In a priori methods, the decision maker first specifies preferences and hopes and after that a solution satisfying these hopes as well as possible is identified. The drawback with this kind of methods is that the decision maker may have too optimistic hopes and the final outcome may be a disappointment or it may be otherwise difficult for the decision maker to specify hopes in advance. On the other hand, in a posteriori methods a representation of the set of nondominated solution is first generated and then the decision maker is supposed to select the most satisfactory solution. The difficulty here is how to display the large amount of data to the decision maker and how to support her/him in finding the best of them. Furthermore, generating a large set of nondominated solutions may be computationally expensive for complicated real-life problems. A possibility to overcome the above-mentioned weaknesses is to use interactive methods, where the decision maker actively takes part in the iterative solution process and specifies preference information gradually. In this way, the decision maker can learn about the problem, its possibilities and limitations as well as the interdependencies among the criteria and possibly even change one's mind about what is desirable. Because only such nondominated solutions are generated that are interesting to the decision maker, interactive methods are computationally much less expensive than, for example, a posteriori methods.

When the decision maker iteratively directs the search for the most satisfactory solution in interactive methods, it may be advisable to first get a rough overview of what the nondominated set looks like, in other words, what kind of compromises are feasible. After that, it is easier to specify preferences. An example of such an approach where a visualization tool based on interactive decision maps (see Lotov *et al.* (2004)) and an interactive classification-based NIMBUS method (see Miettinen and Mäkelä (1999), Miettinen (1999), Miettinen and Mäkelä (2000) and Miettinen and Mäkelä (2006)) are hybridized is given in Miettinen *et al.* (2003). However, that kind of an approach is best suited for problems with less than five criteria.

In this paper, we present a new interactive hybrid approach for solving multicriteria optimization problems. First we produce a rough approximation of the nondominated set and then we let the decision maker indicate where to refine the approximation. In this way, (s)he iteratively directs the search towards the best nondominated solution. Piecewise linear approximations that we use provide a powerful and efficient tool to give an overview of what the set of nondominated solutions looks like. When the decision maker specifies least acceptable values

for different criteria in the form of a reference point, (s)he can zoom in or out in the nondominated set and locate the most satisfactory region. Finally, the selected solution can be given as a starting point to some interactive method for fine-tuning, if desired.

The rest of this paper is organized as follows. In Section 2 we give a problem formulation and introduce the main concepts used. Piecewise linear approximations of the nondominated set are briefly introduced in Section 3 and Section 4 is devoted to the new interactive approach as well as discussion related to it. The issue of updating the reference point is considered in Section 5. Finally, the paper is concluded in Section 6.

2 Problem Formulation

The following notation is used throughout the paper.

We denote components of vectors by subscripts and enumerate vectors by superscripts. Let $u, w \in \mathbb{R}^n$ be two vectors. The notation $u > w$ means that $u_i > w_i$ for all $i = 1, \dots, n$ whereas $u \geq w$ denotes that $u_i \geq w_i$ for all $i = 1, \dots, n$, but $u \neq w$. On the other hand, the notation $u \geq w$ allows equality. The symbols $<, \leq, \leq$ are used accordingly. We denote the non-positive orthant by $\mathbb{R}_{\leq}^n := \{x \in \mathbb{R}^n : x \leq 0\}$. The set \mathbb{R}_{\geq}^n is defined accordingly and the set $\{u\} + \mathbb{R}_{\geq}^n$, where $u \in \mathbb{R}^n$, is referred to as a dominating cone.

We consider the following general multicriteria optimization problem

$$\begin{aligned} & \max \quad \{z_1 = z_1(x)\} \\ & \quad \vdots \\ & \max \quad \{z_n = z_n(x)\} \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^m$ is the *feasible set* and $z_i(x), i = 1, \dots, n$, are real-valued functions. We define the *set of all feasible criterion vectors* Z , the *set of all (globally) nondominated criterion vectors* N and the *set of all efficient points* E of (1) as follows

$$\begin{aligned} Z &= \{z \in \mathbb{R}^n : z = z(x), x \in X\} = z(X) \\ N &= \{z \in Z : \text{there exists no } \tilde{z} \in Z \text{ such that } \tilde{z} \geq z\} \\ E &= \{x \in X : z(x) \in N\}, \end{aligned}$$

where $z(x) = (z_1(x), \dots, z_n(x))^T$. We assume that the set Z is \mathbb{R}_{\leq}^n -closed, that is, the set $Z + \mathbb{R}_{\leq}^n$ is closed. In what follows, we use the notation $\bar{Z}_{\leq} = Z + \mathbb{R}_{\leq}^n$.

The set of properly nondominated solutions is defined according to Geoffrion (1968): A point $\bar{z} \in N$ is called *properly nondominated*, if there exists $M > 0$

such that for each $i = 1, \dots, n$ and each $z \in Z$ satisfying $z_i > \bar{z}_i$ there exists a $j \neq i$ with $z_j < \bar{z}_j$ and

$$\frac{\bar{z}_i - z_i}{z_j - \bar{z}_j} \leq M.$$

Otherwise $\bar{z} \in N$ is called *improperly nondominated*. Moreover, a point $\bar{z} \in Z$ is called *weakly nondominated* if there does not exist $z \in Z$ with $z < \bar{z}$.

The point $z^* \in \mathbb{R}^n$ with

$$z_i^* = \max\{z_i(x) : x \in X\} + \epsilon_i \quad i = 1, \dots, n$$

is called the *ideal (utopia) criterion vector*, where the components of $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}^n$ are small positive numbers. We assume that we can find $u \in \mathbb{R}^n$ such that $u + Z \subseteq \mathbb{R}_{\leq}^n$ and, thus, an ideal criterion vector exists. Without loss of generality we assume that $z^* = 0$. For bicriteria problems, the point in \mathbb{R}^2 with components

$$\max\left\{z_i(\bar{x}) : z_j(\bar{x}) = \max_{x \in X} z_j(x), j \neq i\right\} \quad i = 1, 2$$

is called the *nadir point*. Note that this definition cannot be directly generalized to multicriteria problems with $n > 2$.

By a *reference point* we mean a point in the criterion space \mathbb{R}^n . Note that it does not have to be in Z . Typically, it consists of desirable or acceptable criterion values for each criterion and is specified by a *decision maker*, a person who knows the problem considered well and is able to specify preference information related to it.

3 Approximation of the Nondominated Set

Iteratively improved piecewise linear approximations of the nondominated set can be a powerful tool to simultaneously obtain an overview of the alternatives available and a representation of candidate solutions of a multicriteria optimization problem (1) in an efficient way. A recent survey of different approximation approaches in multicriteria optimization is given in Ruzika and Wiecek (2005).

In the next section we shall show that a piecewise linear approximation can serve as the basis for a further refinement of the decision maker's preferences, which can then be used to identify preference regions in which the approximation has to be refined. We suggest the application of an adaptive approximation procedure as described in Klamroth *et al.* (2002) since it is particularly well-suited for guided refinements based on shifting and moving reference points, and can thus be nicely embedded in interactive algorithms.

Let us next describe the global approximation procedure in brief for convex problems. The method suggested in Klamroth *et al.* (2002) generates piecewise

linear approximations in a problem-dependent way, utilizing polyhedral distance functions to construct the approximation and evaluate its quality. The functions automatically adapt to the problem structure and scaling, and thus quickly adapt to the currently investigated region of the nondominated set. This makes the approximation phase unbiased and self-driven.

Let Z be \mathbb{R}_{\geq}^n -convex (i.e., $Z + \mathbb{R}_{\geq}^n$ is convex) with $\text{int } Z \neq \emptyset$, and first suppose that a reference point \hat{z}^0 is given that satisfies $N \subseteq \hat{z}^0 + \mathbb{R}_{\geq}^n$. This assumption implies that the complete nondominated set N can be generated with the suggested method, starting with this reference point. For a polyhedral gauge $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$, consider the problem

$$\begin{aligned} \max \quad & \gamma(z(x) - \hat{z}^0) \\ \text{s.t.} \quad & z(x) \in \hat{z}^0 + \mathbb{R}_{\geq}^n \\ & x \in X. \end{aligned} \tag{2}$$

If γ is defined by a symmetrical unit ball B (i.e., $\gamma(x) := \inf\{\lambda \in \mathbb{R} : x \in \lambda B\}$) centered at the origin (which can be assumed to coincide with the reference point), that is obtained by symmetrically extending a given piecewise linear inner approximation of N in the criterion space (see Figure 1 for an example), problem (2) finds a feasible point $z(\bar{x}) \in Z$ that maximizes the problem-dependent γ -distance from the current approximation in the criterion space. Schandl *et al.* (2002a) showed that such norms γ are oblique norms, that is, they are absolute (in other words, $\gamma(w) = \gamma(u)$ for all $w \in R(u) := \{w \in \mathbb{R}^n : |w_i| = |u_i| \text{ for all } i = 1, \dots, n\}$) and they satisfy $(z - \mathbb{R}_{\geq}^n) \cap \mathbb{R}_{\geq}^n \cap \partial B = \{z\}$ for all $z \in (\partial B)_{\geq}$.

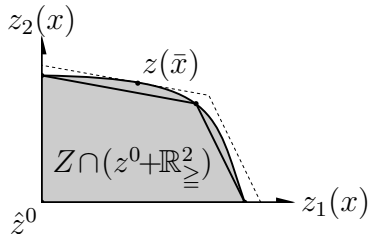


Figure 1: Illustration of an oblique norm applied in problem (2) to an example problem with two criteria

According to the following theorem we know that the solution of problem (2) always gives nondominated solutions, and any properly nondominated solution can be found for convex problems.

Theorem 1 (Schandl *et al.* (2002b)).

- (i) If γ is an oblique norm with reference point \hat{z}^0 satisfying $N \subseteq \hat{z}^0 + \mathbb{R}_{\geq}^n$, then every optimal solution \bar{x} of (2) is an efficient solution of the multicriteria optimization problem (1).
- (ii) If Z is \mathbb{R}_{\geq}^n -convex and if \bar{x} is a properly efficient solution of (1), then there exists an oblique norm γ with reference point \hat{z}^0 satisfying $N \subseteq \hat{z}^0 + \mathbb{R}_{\geq}^n$ such that \bar{x} solves (2).

Note that while part (i) of Theorem 1 remains true for any reference point $\hat{z}^0 \in Z_{\leq}$, statement (ii) must be weakened in this case. In general, only those parts of N located in $\hat{z}^0 + \mathbb{R}_{\geq}^n$ can be generated from a fixed reference point by varying the unit ball B of the oblique norm γ :

Corollary 1.

- (i) If γ is an oblique norm with reference point $\hat{z}^0 \in Z_{\leq}$, then every optimal solution \bar{x} of (2) is an efficient solution of (1).
- (ii) If Z is \mathbb{R}_{\geq}^n -convex and if \bar{x} is a properly efficient solution of (1) in $\hat{z}^0 + \mathbb{R}_{\geq}^n$, then there exists an oblique norm γ with reference point \hat{z}^0 such that \bar{x} solves (2).

Let $d^1, \dots, d^s \in \mathbb{R}^n$ be the normal vectors of the facets of the unit ball B of a polyhedral gauge γ such that $\{z \geq 0 : d^j z \leq 1, j = 1, \dots, s\} = B \cap \mathbb{R}_{\geq}^n$ and $\{z \geq 0 : d^j(z - \hat{z}^0) \leq 1, j = 1, \dots, s\} \subseteq Z_{\leq} = Z + \mathbb{R}_{\geq}^n$. Then problem (2) can be formulated as the following disjunctive programming problem:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \bigvee_{j=1}^s (d^j(z(x) - \hat{z}^0) \geq \lambda \wedge x \in X) \\ & \lambda \in \mathbb{R}. \end{aligned} \tag{3}$$

Figure 2 shows an example with two facets represented by the normal vectors d^1 and d^2 . The point $z(\bar{x})$ corresponds to an optimal λ in (3).

Problem (3) can be decomposed into multiple subproblems, each of a particularly simple structure. For this purpose, let B be the unit ball of γ and denote by C_1, \dots, C_s the fundamental cones of $B \cap \mathbb{R}_{\geq}^n$. If d^j is the normal vector of the facet of the cone C_j , $j = 1, \dots, s$, then (2) can be decomposed into s subproblems (P^j) , $j = 1, \dots, s$, of the form

$$\begin{aligned} \lambda_j = \max \quad & d^j(z(x) - \hat{z}^0) = \sum_{i=1}^n d_i^j (z_i(x) - \hat{z}_i^0) \\ \text{s.t.} \quad & x \in X \end{aligned} \tag{4}$$

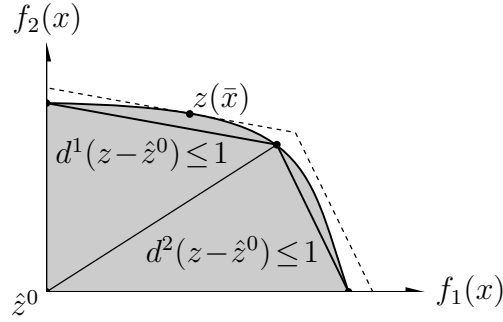


Figure 2: Illustration of problem (3) in a case with two criteria

from which the maximum value of λ_j , $j = 1, \dots, s$ must be selected to obtain an overall optimal solution of (3). Note that each subproblem (4) corresponds to a weighted-sums scalarization of the multicriteria optimization problem (1) and contains only the problem dependent constraints $x \in X$.

The approximation algorithm suggested in Klamroth *et al.* (2002) now iteratively solves problem (2) by computing optimal solutions of all newly generated subproblems of the form (4), starting from an initial approximation that can be generated, for example, by solving $\min\{z_i(x) : x \in X\}$ for all $i = 1, \dots, n$.

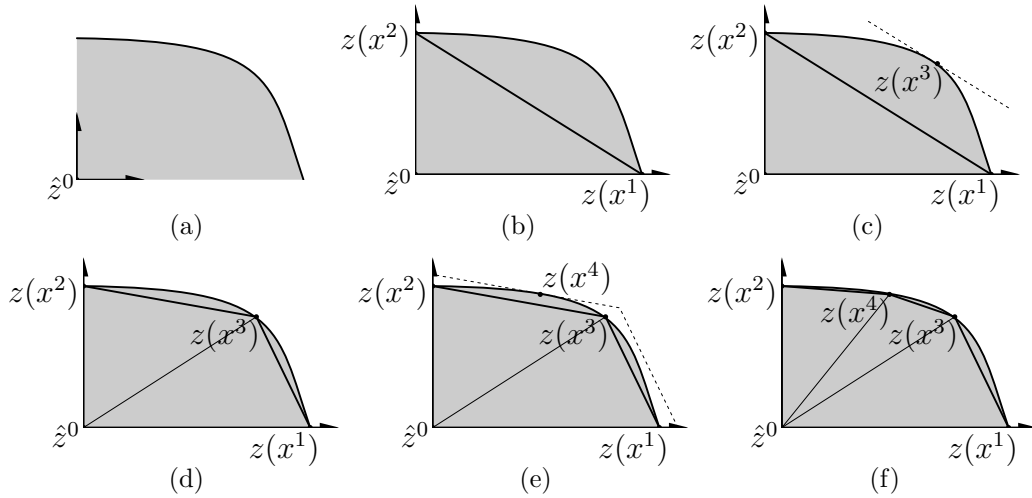


Figure 3: Inner approximation algorithm

In each iteration, the point of “worst” approximation is added to the current approximation which leads to an adaptive update of the polyhedral gauge γ and thus to the generation of a new set of “active” subproblems (4) in the updated cones. Figure 3 illustrates the procedure with an example of the inner approximation for a convex problem. Outer approximations can be constructed in a similar way.

Note that in the bicriteria case each iteration of the approximation algorithm involves the solution of only two weighted-sums scalarizations (4) of problem (1).

Even though the number of active subproblems per iteration may theoretically be larger for $n \geq 3$, it can be expected not to have a considerable impact on the average time needed to find the next iterate.

General convergence results based on previous work on the approximation of convex bodies by Rote (1992), Kamenev (1992), Kamenev (1994) and Lotov *et al.* (2004) yield:

Theorem 2 (Klamroth and Tind (2005)). *Consider a multicriteria optimization problem (1) with n criteria, $n \geq 2$. Then the approximation error after m iterations of the approximation algorithm described above, measured by the adaptive polyhedral gauge γ , and started from a reference point \hat{z}^0 such that $N \subseteq \hat{z}^0 + \mathbb{R}_{\geq}^n$, decreases by the order of $O(m^{-\frac{2}{n-1}})$ which is optimal.*

4 Interactive Approach with Reference Points

In Section 3, we assumed that the reference point $\hat{z}^0 \in \mathbb{R}^n$ was fixed and satisfied the assumption $N \subseteq \hat{z}^0 + \mathbb{R}_{\geq}^n$. But if we allow the reference point to change, for example, by moving the reference point closer to a region of interest for the decision maker, the approximation can be refined in specific areas identified by the decision maker. In this way we get an interactive procedure. It is important that we can decrease computational costs because the approximation is made more accurate only in those regions the decision maker is interested in.

4.1 The Impact of the Reference Point

By allowing the decision maker to specify the reference point, it can be used to

- capture preference information of the decision maker in the form of minimal criterion values that should be attained,
- and zoom the approximation in the region of interest to the decision maker.

Since the reference point has an immediate impact on the relative size of the unit ball of the approximating gauge γ , moving \hat{z}^0 closer to the nondominated set will lead to a relative increase of the distance of the approximation from the nondominated set in this area.

In order to obtain meaningful problem formulations (2), the choice of the reference point should be restricted to the set Z_{\leq} . We can distinguish three different situations, where we denote the reference point of the current iteration of the decision process by \hat{z}^i , and that of the next iteration by \hat{z}^{i+1} :

- $\hat{z}^i \leq \hat{z}^{i+1}$: In this case the decision maker has identified a region of interest from the previous approximation and wishes to refine the approximation in the area defined by $(\hat{z}^{i+1} + \mathbb{R}_{\geq}^n) \cap Z_{\leq}$ (zooming in).

- $\hat{z}^i \geq \hat{z}^{i+1}$: The decision maker wishes to explore a larger region surrounding the previously approximated part of the nondominated set (zooming out).
- Otherwise, the decision maker prefers to explore a (partially) different region of the nondominated set without using the information gathered in the previous iterations.

Because the decision maker can freely study different nondominated solutions and get to know the problem, it may be useful to save interesting solutions for later consideration during the search process. This may be particularly useful if the decision maker wants to explore different regions of the nondominated set as mentioned above in the third item.

4.2 Interactive Algorithm

Let us denote the set of saved solutions by A . The main steps of the interactive algorithm can be given as follows:

1. Set $i = 0$ and $A = \emptyset$. Ask the decision maker to specify the number of solutions P to be shown to her/him at each iteration.
2. Ask the decision maker to select a reference point $\hat{z}^i \in Z_{\leq}$.
3. Construct a (rough) approximation of $(\hat{z}^i + \mathbb{R}_{\geq}^n) \cap Z_{\leq}$ consisting of P solutions, that is, criterion vectors.
4. Show the approximation to the decision maker. If the decision maker wishes to save some of the solutions, add them in A .
5. If some of the points in the approximation or in A is acceptable and desirable for the decision maker as a final solution, stop. Otherwise, if the decision maker would like to continue with a reference point or classification based interactive method in the current neighborhood, take the current reference point as bounds for the criteria and continue with another method. Otherwise, go to step (2) and let the decision maker update P if so desired.

Let us next discuss some of the steps in the algorithm. In the first iteration, it is possible to use the worst possible criterion values as the components of the reference point so that the decision maker gets a rough approximation of the whole nondominated set to start with. For this purpose, the criterion values calculated as $\min\{z_i(x) : x \in X\}$, for all $i = 1, \dots, n$, can be given to her/him. Alternatively, the nadir point can be used if it is available.

It is up to the decision maker how many solutions the approximation should contain. The more solutions there are, the more accurate the approximation gets but, on the other hand, the more difficult it is for the decision maker to

study them. The decision maker can also change the number of solutions to be calculated in the approximation during the solution process if so desired. Alternatively, if the decision maker does not want to set the number of solutions in the approximation, another condition based on the relative approximation error can be specified as well. (Note that, according to Theorem 2, the number of solutions in the approximation also induces a bound on the approximation error.)

Different visualization tools can be used to help the decision maker in comparing the different solutions in the approximation. Examples of visualizations for four nondominated solutions involving three criteria are given in Figure 4. This figure illustrates bars charts, bars in three dimensions, value paths, spider web charts and petal diagrams (see, for example, Miettinen (1999), Miettinen (2003) and Trinkaas and Hanne (2005)). By having different options available the decision maker may choose which types of visualizations (s)he finds most informative.

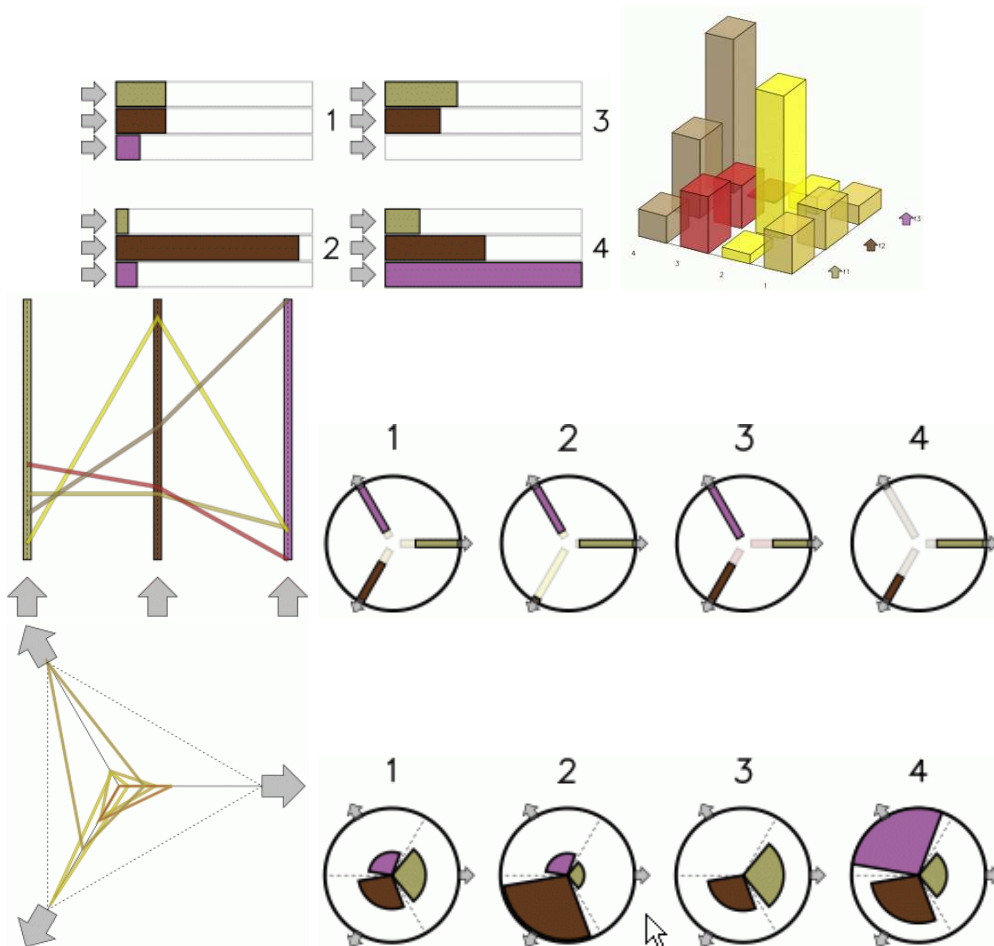


Figure 4: Different visualization possibilities

It is also possible to arrange the solutions in a decreasing order according to some of the criteria in order to help the decision maker to find the most interesting ones. When studying the different solutions, the decision maker can drop uninteresting solutions and filter out some of the solutions by specifying upper or lower bounds or interesting intervals for criterion values. With the same visualization tools the decision maker can also study the set A where (s)he has saved interesting solutions.

After the decision maker has identified a desirable region in the set of non-dominated solutions, (s)he can select one of the points in the last approximation as the final solution. Alternatively, (s)he can start an interactive reference point or classification based method so that the bounds specified in the reference point of the last approximation are included as constraints in the problem. In this case, the decision maker can work with the real problem and not the approximation to fine-tune the solution by specifying her/his hopes in the form of a reference point consisting of desirable values for each criterion or in the form of a classification indicating what kind of changes would make the current solution even better. Thanks to the extra constraints set on the criterion values, the consideration now concentrates on the region found interesting in the preceding phase done with approximations. Note that in this phase, the reference point can be any point in the criterion space. Examples of reference point based methods include the reference point method by Wierzbicki (1986) and reference direction approach by Korhonen and Laakso (1986) whereas classification based methods include the satisficing trade-off method by Nakayama (1995) and NIMBUS method by Miettinen (1999) and Miettinen and Mäkelä (2006).

5 Updating the Reference Point

After having studied the current approximation, the decision maker may have in mind values of the different criteria (s)he wants to use as the components of the next reference point. If the decision maker wants to zoom in, in other words, wants to specify a reference point closer to the nondominated set, it is possible to show the current reference point to the decision maker and then (s)he can indicate for which criteria and how much higher new values would be acceptable to form the next reference point. Correspondingly, zooming out can be carried out by specifying lower values than in the current reference point. Naturally, the decision maker can specify any criterion values as the components of the next reference point irrespectively what the current reference point is. But if the decision maker wants to zoom in or zoom out in the nondominated set, there are different possibilities how to support the decision maker in specifying the next reference point, that is, new minimal acceptable values for the criteria.

5.1 Guiding the Search by Selecting Nondominated Solutions

One possible way to help the decision maker in specifying a new reference point is to present to her/him a reasonably small set of candidate points as a new reference point. For this purpose, suppose that the current approximation with the reference point $\hat{z}^i \in Z_{\leq}$ is defined by its normal vectors $d^1, \dots, d^s \in \mathbb{R}^n$, and the approximation consists of the solutions $z(x^1), \dots, z(x^P) \in N$. Appropriate candidates for the next reference point $\hat{z}^{i+1} \in Z_{\leq}$ would then, for example, be the points

$$z^{(i+1),j} := \hat{z}^i + \beta \cdot (z(x^j) - \hat{z}^i), \quad j = 1, \dots, P \quad (5)$$

with a refinement parameter $\beta \in (0, 1)$ that may also be selected by the decision maker. Note that by setting β to be positive, we move closer to the nondominated set, that is, zoom in. On the other hand, we can zoom out by using a negative value.

Let us assume that the decision maker has identified the most satisfactory solution $z(x^j)$ among the ones in the current approximation, that is, $z(x^1), \dots, z(x^P) \in N$. Then, in order to further explore the (variable-size) neighborhood of this solution, the point $z^{(i+1),j}$ could be selected to serve as the reference point \hat{z}^{i+1} for the next iteration of the procedure.

Let us next assume that the decision maker wants to zoom in. Once the new reference point has been chosen, the approximation is filtered to leave in the approximation only the solutions located in the new approximation area given by $(\hat{z}^{i+1} + \mathbb{R}_{\geq}^n) \cap Z_{\leq}$ such that the respective extremal solutions (maximizing the individual criteria in the reduced feasible set) become a part of the approximation. This can be realized, for example, by performing a direction search according to Pascoletti and Serafini (1984) along the coordinate directions e^j , $j = 1, \dots, n$, starting at the new reference point \hat{z}^{i+1} :

$$\begin{aligned} \text{lexmax} \quad & (\alpha, \sum_{j=1}^n q_j) \\ \text{s.t.} \quad & z(x) = \hat{z}^{i+1} + \alpha e^j + q \\ & q \in \mathbb{R}_{\geq}^n \\ & x \in X, \end{aligned} \quad (6)$$

where the lexicographic maximization includes the term $\sum_{j=1}^n q_i$ as a second function to be optimized to avoid weakly nondominated solutions.

If necessary, that is, if the number of solutions in the resulting approximation is below the pre-specified number P , the approximation is then refined by calculating a new approximation with reference point \hat{z}^{i+1} . In this case, the solutions of the existing approximation are not deleted but new solutions are just included in it to get a new approximation. Given the next approximation, the refinement process is continued as described in the interactive algorithm in Subsection 4.2.

On the other hand, if the decision maker wants to zoom out by specifying a negative β in (5) and wants to keep the number of solutions in the approximation equal to pre-specified P , then (s)he has to delete some of the solutions in the existing approximation, for example, with the help of the visualization tools. After that, as many new solutions are added to the approximation with the new reference point. On the other hand, the decision maker can increase P and then as many new points are added to the approximation as needed. If so desired, the individually maximized solutions can be added in the approximation as in the case of zooming in.

The search may be diversified by storing all candidates for possible reference points from the previous iterations and allowing the decision maker to select one of them (at certain stages of the procedure) as the next reference point.

Figure 5 shows an example of the above-described ideas of generating a new reference point. This zooming in process is related to the problem introduced in Figure 3. We set $P = 4$, in other words, we assume that the approximation is sufficiently fine as soon as it has four solutions. Given the initial approximation of Figure 3(f) obtained with the nadir point \hat{z}^0 used as a reference point, suppose that the point $z(x^3)$ is chosen by the decision maker as the most preferred solution among the generated nondominated points $z(x^1), \dots, z(x^4) \in N$. Setting $\beta = \frac{1}{2}$, the reference point for the next stage of the procedure is $\hat{z}^1 = \hat{z}^0 + \frac{1}{2}(z(x^3) - \hat{z}^0)$, see Figure 5(b). The approximation is updated in the resulting approximation area (see Figure 5(c)), and since it still has four solutions and is thus sufficiently fine, a new reference point is requested from the decision maker. Selecting again $z(x^3)$ as the most preferred solution and setting $\beta = \frac{1}{2}$ leads to the next reference point $\hat{z}^2 = \hat{z}^1 + \frac{1}{2}(z(x^3) - \hat{z}^1)$ (see Figure 5(d)), from which the approximation is again updated (see Figure 5(e)) and refined (see Figure 5(f)).

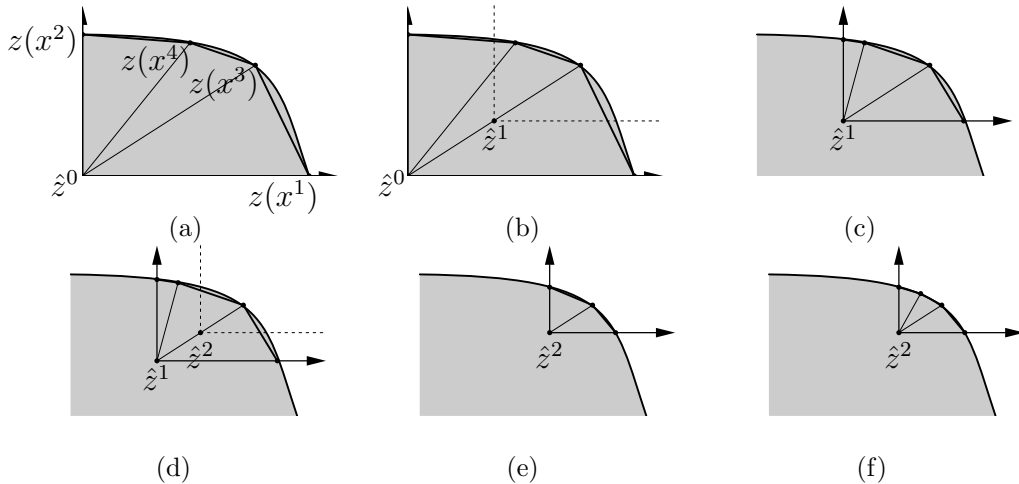


Figure 5: Updating the reference point using candidate reference points

The final approximation obtained in Figure 5(f) is shown in Figure 6 without

the nondominated set (which is actually unknown in real problems).

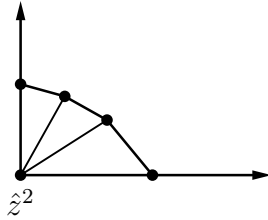


Figure 6: Obtained approximation after two iterations

5.2 Guiding the Search by Selecting Lower Bounds

Using bounds to refine the search for a most preferred solutions was already suggested by Fandel (1972) and will here be embedded into a sequential refinement of approximations as described in Subsection 4.2.

If the decision maker wants to zoom in, as an alternative to the approach described in the previous subsection, the decision maker could be asked to give an order for the criteria, for example, according to how interesting they are (which may differ from iteration to iteration). Then the components of the next reference point are specified in this order.

Let us assume that the ranking corresponds to the numbering of the criteria. Then the solutions $z(x^1), \dots, z(x^P)$ forming the current approximation are presented to the decision maker, and (s)he selects from these points some solution $z(x^j)$, $j \in \{1, \dots, P\}$ to fix the lower bound as $b_1 := z_1(x^j)$ for the first criterion. The next bound (for the second criterion) is selected using a subset of the solutions in the current approximation. This subset is of the form

$$\{z(x^k) : z_1(x^k) \geq b_1, k = 1, \dots, P\}, \quad (7)$$

and after having a look at it the decision maker selects the lower bound for the second criterion. This procedure is repeated until lower bounds b_i have been obtained for all $i = 1, \dots, n$. Smaller and smaller subsets corresponding to (7) are shown to the decision maker in order to ensure that the obtained bounds are feasible in the sense that the vector $b = (b_1, \dots, b_n)^T$ lies in Z_{\leq} . If the subset to be shown decreases too fast, the decision maker can take a step backward and select a looser value for the previous criterion.

Consequently, the new reference point from which the approximation is recomputed and refined (c.f. Subsection 5.1) is now given by $\hat{z}^{i+1} := (b_1, \dots, b_n)^T \in Z_{\leq}$. Figure 7 illustrates the selection of a new reference point and consecutive refinements of the approximation when the current approximation and its subsets are used to get bounds for the new reference point.

The obtained approximation (without the nondominated set in the background) is depicted in Figure 8.

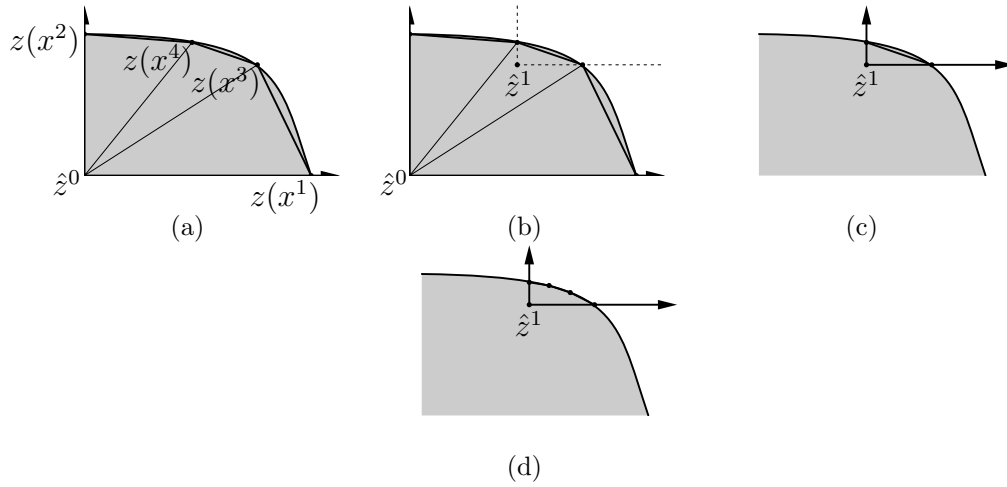


Figure 7: Updating the reference point based on bounds

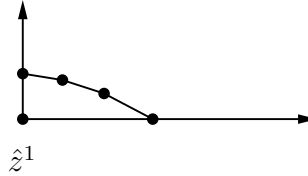


Figure 8: Obtained approximation after one iteration

5.3 Guiding the Search Using Classification

One more possibility to support the decision maker in specifying the next reference point is to use classification. This means that the current reference point is shown to the decision maker and (s)he is asked to classify the criteria into those where the current minimal acceptable level should be

- increased from the current one,
- decreased from the current one or
- kept the same.

Besides classification, the decision maker is also supposed to set the new levels in the first two classes. In order to avoid infeasibility, this choice should be constrained such that the new reference point still satisfies $\hat{z}^{i+1} \in Z_{\leq}$. For example, the desired improvement in the respective criteria could be approximated

by moving a steplength λ towards the corresponding criteria values of a feasible solution of the current approximation (c.f. Subsection 5.1). At the same time, the reference point may be moved away by a step length of $\bar{\lambda}$ from another feasible solution.

The next reference point is then formed using the new as well as the current values (if the last class is also used) for the appropriate criteria. This resembles the classification used in interactive methods for directing the search for the most satisfactory solution see, for example, Miettinen (1999) and Miettinen and Mäkelä (2006).

When the new reference point has been found, the current approximation is once again filtered so that only the solutions in the new approximation area $(\hat{z}^{i+1} + \mathbb{R}_{\geq}^n) \cap Z_{\leq}$ are retained. Using only the first and the last class means zooming in and using the last two classes means zooming out and in those cases recomputing and refining the approximation can be done as described in Subsection 5.1. Otherwise, if the new approximation area does not contain any of the solutions of the current approximation the whole approximation is to be recomputed (with the new reference point). Figures 9 and 10 show an example of the application of this approach.

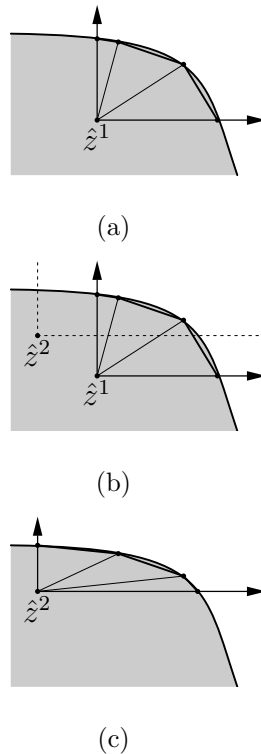


Figure 9: Updating the reference point based on classification

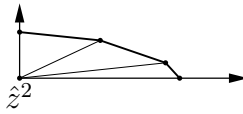


Figure 10: Obtained approximation after one iteration

6 Conclusions

We have introduced a new interactive multicriteria optimization method for convex problems where rough approximations of the nondominated set and reference points specified by a decision maker are combined. In this new approach we hybridize ideas of a posteriori and interactive methods. The advantage of this new interactive learning-oriented method is that the decision maker gets an impression of the behavior of the problem with the help of approximations and can identify the most interesting region of criterion values conveniently. The calculations involved with approximations are fast and efficient. After the desirable region has been identified, the decision maker can fine-tune the solution with existing interactive methods that operate with the original multicriteria optimization problem (and not the approximation) so that consideration is restricted to the interesting region found. In this way, the decision maker can find any desirable nondominated solution faster.

In the suggested new interactive algorithm, the number of optimization problems to be solved during the whole solution process can stay relatively small emphasizing the computational efficiency of the approach. This is explained both by the fact that only such parts of the nondominated set are studied more closely that the decision maker is interested in and because the current approximation can be utilized in many cases and it is then only augmented by new solutions. In other words, the approximation does not have to be calculated from scratch but the existing one can be taken as a part of the new approximation. The usability aspect is also important with interactive methods and here the information given to and asked from the decision maker is easily understandable in the form of criterion values. In other words, not too much cognitive burden is set on the decision maker.

Even though the new method has here been formulated for convex problems, it can be generalized for nonconvex problems in a corresponding way. In this case, the weighted-sums scalarization cannot be used as a part of the approximation procedure but, for example, modifications of the weighted Tchebycheff scalarization, as suggested in Klamroth *et al.* (2002), can be adopted.

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