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* University of Jyväskylä,
Dept. of Mathematical Information Technology

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Ideas of Using Trade-off Information in Supporting the Decision Maker in Reference Point Based Interactive Multiobjective Optimization

Jussi Hakanen, Petri Eskelinen [†]

Dept. of Mathematical Information Technology
P.O. Box 35 (Agora)
FI-40014 University of Jyväskylä, Finland
e-mail: jussi.hakanen@mit.jyu.fi

[†] Helsinki School of Economics
P.O. Box 1210
FI-00101 Helsinki, Finland
e-mail: petri.eskelinen@hse.fi

Abstract

In this paper, some ideas for utilizing trade-off information in supporting the decision maker during the interactive solution procedure of reference point based multiobjective optimization are proposed. The aim is to help the decision maker in finding the most preferred solution by reducing the number of iterations of the interactive solution procedure required. Two different visualizations of trade-off information are presented being easy to understand for the decision maker. In our examples we are considering partial trade-off rates obtained with the method of Sakawa and Yano utilizing the KKT multipliers. In order to demonstrate the ideas presented we solve a simple test problem in an automated test framework to simulate the behavior of the decision maker. Promising results were obtained and they show that this kind of approach can be useful in supporting the decision maker.

Keywords: Multiobjective optimization, interactive methods, trade-off rates, decision maker, reference point, classification

1 Introduction

Interactive multiobjective optimization methods aim at finding the most preferred solution for the multiobjective optimization problem in question with the help of iterative interaction between the method and the decision maker (in abbreviation, DM), the user of the method [3, 10, 20, 33, 34]. In multiobjective optimization, there is not necessarily a unique optimal solution in the sense of single objective optimization, but a set of mathematically equivalent compromise solutions called Pareto optimal solutions. To select the best compromise solution within all the Pareto optimal solutions, we need some additional information. The DM is supposed to have some knowledge about the problem in question and to be able to express preference information about the different Pareto optimal solutions.

In this paper, the idea is to support the DM in finding the most preferred solution. In practice, it has been found that DMs are not willing to use too much time (iterations) in the interactive solution procedure although they prefer interactive methods [16]. Therefore, we want to help the DM in the way that finding the most preferred solution will not require too many iterations of an interactive solution procedure. In other words, we do not want to take too much of DM's time. By generating new solution(s) quickly and reducing the number of iterations required, we can further improve the good properties of interactive solution procedures.

There exist already some ways to support the DM during the solution procedure. For example, the bounds for the Pareto optimal set in the objective space, namely the ideal and the nadir objective vectors, can be shown to the DM so that (s)he gets some idea what kind of solutions can be achieved. In addition, different kind of visualizations of the Pareto optimal solutions already obtained can be shown to the DM in order to help in comparing different solutions and in selecting the preferred ones. This kind of support is available, for example, in WWW-NIMBUS [25, 27], that is an implementation of the interactive NIMBUS® method [20, 23, 24, 27], where the DM is able to choose from several different type of visualizations when comparing Pareto optimal solutions obtained [21, 22].

Another kind of approach to support the DM is to use *trade-off information* (see, for example, [3, 33]). The concept of trade-off is used in the context of multiobjective optimization because Pareto optimal solutions are mathematically incomparable and one has to sacrifice in some objective in order to gain in some other objective, and this is called *trading-off*. In this paper, we consider trade-off information related to the problem itself, and it can be computed without using any preference information from the DM. In other words, this kind of trade-off information describes interdependencies between objective functions and how their values change locally with respect to others. Note that we do not consider trade-off information coming from the DM that is usually called *marginal rate of substitution* or *subjective trade-off* [3].

Previously, trade-offs have been mainly used as a part of some multiobjective optimization method (see, for example, [3, 7, 28, 31, 37, 41]). Our aim in this paper is to consider trade-off information only as a supporting tool for the DM and not to use it in the solution method itself. Especially, our purpose is not to develop a new interactive method but to offer supporting tools for the DM who is using already available interactive methods. Our aim is to present an approach which can be utilized with different kind of classification and reference point based interactive methods. In addition, although the theoretical background of trade-off information has been widely studied [1, 3, 6, 7, 9, 12, 13, 18, 19, 26, 30, 31, 32, 33, 38, 39], there have been only few numerical examples how to utilize this kind of information in practice. Therefore, there is not much evidence of practical usefulness of trade-off information as a decision support tool.

Another important issue is how to present trade-off information to the DM. It is not desirable to increase the cognitive burden set on to the DM by showing him/her too much, too detailed, too complicated or too unreliable information. Therefore, we do not show any numerical trade-off information to the DM but, instead, only some directions about how the values of different objective functions could change when moving away from the current Pareto optimal solution. We realize this by describing two different ways of presenting the trade-off information to the DM. These visualizations aim to present trade-off information in a simple way that is easy to interpret and utilize during the interactive soltion procedure.

In this paper, we have a few goals. First of all, we want to obtain supporting trade-off information with as small additional computation as possible. In addition, we want to help the DM getting convinced that at the end of the interactive solution procedure, (s)he has obtained the best possible compromise solution. Furthermore, we want to present the trade-off information to the DM as clearly as possible so that interpreting and utilizing it becomes easier for the DM.

As trade-off information in this paper, we use partial trade-off rates where trade-off between two objective functions is considered at a time. Here, partial trade-of rates are obtained with the method of Sakawa and Yano [32] where they can be computed with the help of the Karush–Kuhn–Tucker multipliers. Several gradient based single objective optimizers produce these multipliers as a byproduct and, therefore, no additional computation is required. To illustrate our ideas we use a reference point method based on achievement scalarizing functions [36]. The reference point method used is a special case of the hyperplane scalarization by Sakawa and Yano [32, 40].

For simplicity, we consider only unconstrained multiobjective optimization problems. However, our approach can also be utilized with constrained problems and in this sense it is possible to extend this study in the future.

This paper is organized as follows. First, in Section 2 some definitions and theory related to trade-offs is presented, and we also describe how trade-off information can be obtained without adding computational cost significantly. In Section 3 we describe the usage of trade-off information in supporting the DM. Section 4 is devoted to numerical examples. Finally, we give some concluding remarks and future research ideas in Section 5.

2 Trade-offs in multiobjective optimization

2.1 Multiobjective optimization problem

In this paper, we consider nonlinear multiobjective optimization problems of the form

$$\begin{aligned} &\text{minimize} \quad \{f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})\} \\ &\text{subject to} \quad \boldsymbol{x} \in S. \end{aligned} \tag{1}$$

Problem (1) contains k real-valued continuous *objective functions* $f_i : S \to \mathbb{R}$ to be minimized with respect to the *decision variables* $\boldsymbol{x}$ belonging to the *feasible set* $S \subset \mathbb{R}^n$. We assume, that the objective functions f_i, $i = 1, \ldots, k$ are nonlinear and twice continuously differentiable, that is, their derivatives and second derivatives with respect to $\boldsymbol{x}$ are continuous functions. We denote objective function values at $\boldsymbol{x}$ by an *objective vector* $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x}))^T$ belonging to the *objective space* $\mathbb{R}^k$. Note that if the objective function f_i is to be maximized then it is equivalent to minimize the function $-f_i$.

In multiobjective optimization there does not necessarily exist such a unique $\boldsymbol{x} \in S$ that minimizes all the objective functions at the same time but, instead, we must consider a set of optimal solutions. An optimal solution of problem (1) is called a Pareto optimal (efficient, non-dominated) solution:

Definition 2.1
A solution $\hat{\boldsymbol{x}} \in S$ is Pareto optimal *if there exists no other solution $\boldsymbol{x} \in S$ such that $f_i(\boldsymbol{x}) \leq f_i(\hat{\boldsymbol{x}})$ for all $i = 1, \ldots, k$ and at least one of the inequalities is strict. The objective vector $\hat{\boldsymbol{z}}$ is Pareto optimal if the corresponding solution $\hat{\boldsymbol{x}}$ is Pareto optimal.*

Definition 2.1 is for global Pareto optimality. The definition for local Pareto optimality is the following:

Definition 2.2
A solution $\hat{\boldsymbol{x}} \in S$ is locally Pareto optimal *if there exists a neighborhood $N(\hat{\boldsymbol{x}})$ of $\hat{\boldsymbol{x}}$ such that $\hat{\boldsymbol{x}}$ is Pareto optimal in $N(\hat{\boldsymbol{x}}) \cap S$. The objective vector $\boldsymbol{f}(\hat{\boldsymbol{x}})$ is locally Pareto optimal if the corresponding solution $\hat{\boldsymbol{x}}$ is locally Pareto optimal.*

In what follows we also use the concept of weak Pareto optimality:

Definition 2.3
A solution $\hat{\boldsymbol{x}} \in S$ is weakly Pareto optimal *if there exists no other solution $\boldsymbol{x} \in S$ such that $f_i(\boldsymbol{x}) < f_i(\hat{\boldsymbol{x}})$ for all $i = 1, \ldots, k$. The objective vector $\hat{\boldsymbol{z}}$ is weakly Pareto optimal if the corresponding solution $\hat{\boldsymbol{x}}$ is weakly Pareto optimal.*

The set containing all the Pareto optimal solutions related to the problem (1) is called the *Pareto optimal set* and it is denoted by $E \subset S$. If $\boldsymbol{x}^{*,i}$ is a solution of problem $\min_{\boldsymbol{x} \in S} f_i(\boldsymbol{x})$ and $\boldsymbol{x}^{\text{nad,i}}$ is a solution of problem $\max_{\boldsymbol{x} \in E} f_i(\boldsymbol{x})$ then vectors $\boldsymbol{z}^* = (f_1(\boldsymbol{x}^{*,1}), \ldots, f_k(\boldsymbol{x}^{*,k}))^T$ and $\boldsymbol{z}^{\text{nad}} = (f_1(\boldsymbol{x}^{\text{nad,1}}), \ldots, f_k(\boldsymbol{x}^{\text{nad,k}}))^T$ are called the *ideal* and *nadir objective vectors*, respectively. In other words, the ideal and the nadir objective vectors are the lower and the upper bounds for objective function values in the Pareto optimal set E, respectively. The nadir objective vector can not usually be computed exactly and, therefore, it needs to be approximated. A widely used though unreliable approximation can be obtained by using the *payoff table* [20]. In order to avoid computational difficulties, a *utopian objective vector* $\boldsymbol{z}^{**}$ is usually used instead of the ideal objective vector. The utopian objective vector can be defined, for example, by $z_i^{**} = z_i^* - \epsilon$ for each $i = 1, \ldots, k$, where ϵ is a small strictly positive real number.

2.2 Simplified reference point method

There exist various interactive methods that can be used to solve the multiobjective optimization problem (1). In this paper, we concentrate on reference point based methods and we use a simple variation of the achievement scalarizing function based on the reference point method proposed by Wierzbicki [36] for demonstration purposes. We use the simplified reference point method as an interactive method and it proceeds as follows.

At each new iteration the DM defines desirable *aspiration levels* $\bar{z}_i \in [z_i^*, z_i^{\text{nad}}]$ for each objective function f_i, $i = 1, \ldots, k$. The aspiration levels constitute a *reference point* $\bar{\boldsymbol{z}} = (\bar{z}_1, \ldots, \bar{z}_k)^T \in \mathbb{R}^k$ and they represent the desirable values of the objective functions for the DM. The given reference point $\bar{\boldsymbol{z}}$ is then projected to the Pareto optimal set by solving the following problem

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \max_{i=1,\ldots,k} [w_i(f_i(\boldsymbol{x}) - \bar{z}_i)] \\ \text{subject to} \quad & \boldsymbol{x} \in S, \end{aligned} \tag{2}$$

where $w_i > 0$, $i = 1, \ldots, k$ are the fixed weighting coefficients used for scaling. It can be shown, that a solution of problem (2) is at least weakly Pareto optimal [20].

Note that problem (2) is in the general case nonsmooth independent of the properties of the functions in the problem. However, if the functions in the problem are differentiable, we can formulate a smooth variant of problem (2) which enables the usage of gradient based optimizers:

$$\begin{aligned} \underset{\boldsymbol{x},\delta}{\text{minimize}} \quad & \delta \\ \text{subject to} \quad & w_i(f_i(\boldsymbol{x}) - \bar{z}_i) \le \delta, \quad i = 1, \ldots, k, \\ & \boldsymbol{x} \in S, \quad \delta \in \mathbb{R}. \end{aligned} \tag{$P(\bar{\boldsymbol{z}})$}$$

In problem $(P(\bar{\boldsymbol{z}}))$, we have introduced an additional variable $\delta \in \mathbb{R}$ as a new objective function and converted the objective function of problem (2) into k inequality constraints. Note that a solution of problem $(P(\bar{\boldsymbol{z}}))$ is also a solution of problem (2).

In our simplified method a new Pareto optimal solution is produced at every iteration using the reference point given by the DM. The solution procedure is stopped when the DM considers that the most preferred Pareto optimal solution is found. Let us note that even if the solutions obtained using the problem formulation ($P(\bar{\boldsymbol{z}})$) are guaranteed to be only weakly Pareto optimal, it does not actually matter in our presentation. In this context, we use the formulation only to introduce our ideas. Later on it can be extended and more complex formulations can be also considered.

The parameters of problem ($P(\bar{\boldsymbol{z}})$) are the aspiration levels $\bar{z}_i$, $i = 1, \ldots, k$, and different (weakly) Pareto optimal solutions are obtained by changing the reference point. The following theorem can be formulated for the solutions of problem $P(\bar{\boldsymbol{z}})$ [20]:

Theorem 2.4
Let $\bar{\boldsymbol{z}}^* \in \mathbb{R}^k$ be a given reference point, then a solution $\boldsymbol{x}^*$ of problem ($P(\bar{\boldsymbol{z}}^*)$) is weakly Pareto optimal. If the solution $\boldsymbol{x}^*$ is unique, then it is Pareto optimal.

Even if the feasible set is mentioned occasionally in this text we actually consider only unconstrained problems. Everywhere in this presentation we can set $S = \mathbb{R}^n$. Because we are considering unconstrained problems we assume that all the objective functions are bounded below so that for each objective function f_i there exists at least one global minimizer $\boldsymbol{x}^* \in \mathbb{R}^n$ of the objective function f_i.

2.3 Definition of trade-off

Next, we present some concepts related to trade-offs and our presentation follows [6]. The *ratio of change* between points $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ involving objective functions f_i and f_j is defined by

$$T_{ij}(\boldsymbol{x}, \hat{\boldsymbol{x}}) = \frac{f_i(\boldsymbol{x}) - f_i(\hat{\boldsymbol{x}})}{f_j(\boldsymbol{x}) - f_j(\hat{\boldsymbol{x}})}, \quad \boldsymbol{x}, \hat{\boldsymbol{x}} \in S,$$

where $f_j(\boldsymbol{x}) \neq f_j(\hat{\boldsymbol{x}})$. If $f_l(\boldsymbol{x}) = f_l(\hat{\boldsymbol{x}})$ for all $l \neq i, j$ we call T_{ij} *partial trade-off* between points $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$. If $f_l(\boldsymbol{x}) \neq f_l(\hat{\boldsymbol{x}})$ for at least one $l \neq i, j$, then T_{ij} is called *total trade-off* .

Using the ratio of change $T_{ij}(\boldsymbol{x}, \hat{\boldsymbol{x}})$ we can define *total trade-off rate* at the point $\boldsymbol{x} \in \mathbb{R}^n$ to direction $\boldsymbol{d}$ as a limit

$$t_{ij}(\boldsymbol{x}, \boldsymbol{d}) = \lim_{\alpha \searrow 0} T_{ij}(\boldsymbol{x} + \alpha \boldsymbol{d}, \boldsymbol{x}),$$

where we assume that $\boldsymbol{d} \neq \boldsymbol{0}$ is a feasible direction, that is, there exists $\alpha_0 > 0$ such that $\boldsymbol{x} + \alpha \boldsymbol{d} \in S$ for all $\alpha \in [0, \alpha_0)$. If $\boldsymbol{d}$ is a feasible direction such that there exists $\bar{\alpha} > 0$ satisfying $f_l(\boldsymbol{x} + \alpha \boldsymbol{d}) = f_l(\boldsymbol{x})$ for all $l \neq i, j$ and for all $0 \leq \alpha < \bar{\alpha}$, then the corresponding t_{ij} is called a *partial trade-off rate.* Note that in continuously differentiable case the total and partial trade-off rates can equivalently be formulated by (see [6, 20])

$$t_{ij}(\boldsymbol{x}, \boldsymbol{d}) = \frac{\nabla f_i(\boldsymbol{x})^T \boldsymbol{d}}{\nabla f_j(\boldsymbol{x})^T \boldsymbol{d}} \tag{3}$$

and

$$t_{ij}(\boldsymbol{x}) = \frac{\partial f_i(\boldsymbol{x})}{\partial f_j},$$

respectively, where $\boldsymbol{x} \in S$, $\boldsymbol{d}$ is a feasible direction and $\nabla f_j(\boldsymbol{x})^T \boldsymbol{d} \neq 0$. In this presentation, trade-off rate $t_{ij}(\boldsymbol{x})$ without a direction is considered to be a partial trade-off rate.

Figure 1 illustrates the concept of trade-off in objective space in the case of two objective functions. In this figure, the set $Z = \boldsymbol{f}(S)$ denotes the image of the feasible set and the bold line at its boundary indicates the set of Pareto optimal objective vectors $\boldsymbol{z} \in \boldsymbol{f}(E)$, that is, the Pareto optimal set. The trade-off rate related to the objective vector $\boldsymbol{z}$ is depicted by an arrow. This means that if we want to improve objective f_2 by amount Δf_2 we can approximate the impairment in objective f_1 by Δf_1. By looking at Figure 1, it can be clearly seen that a trade-off rate at some point is only a linear approximation and, therefore, can only be used in some finite neighborhood of the point considered. Let us point out that, when two objectives are considered, the partial trade-off rate is always equal to the total trade-off rate.

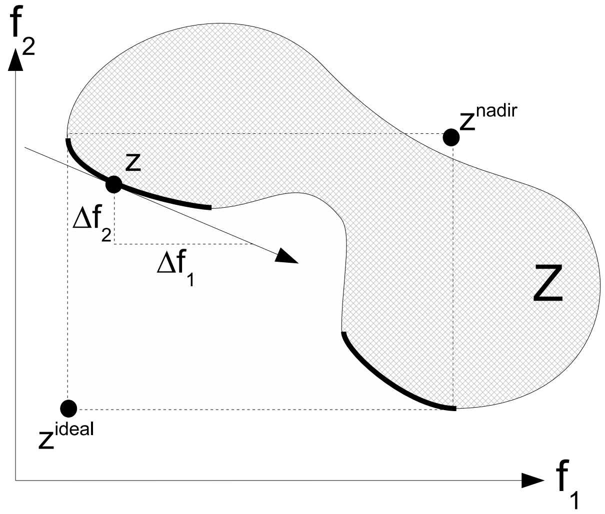

Figure 1: An illustration trade-off for two objectives.

In this paper, we consider only trade-off rates and by trade-off information we mean trade-off rates. In addition, we are interested in trade-off rates only at the Pareto optimal points so that in all the above trade-off definitions a feasible set S is replaced by the Pareto optimal set E. At every point $\boldsymbol{x} \in E$ we can produce the *trade-off rate matrix*

$$M(\boldsymbol{x}, \boldsymbol{d}) = \begin{bmatrix} t_{11}(\boldsymbol{x}, \boldsymbol{d}) & \dots & t_{k1}(\boldsymbol{x}, \boldsymbol{d}) \\ \vdots & \ddots & \vdots \\ t_{1k}(\boldsymbol{x}, \boldsymbol{d}) & \dots & t_{kk}(\boldsymbol{x}, \boldsymbol{d}) \end{bmatrix}$$

which reflects sensitivities between objectives when we are moving to the direction $\boldsymbol{d}$ from the point $\boldsymbol{x}$. A similar matrix $M(\boldsymbol{x})$ can be defined also for partial trade-off rates. Then, it has only $\boldsymbol{x}$ as a parameter and the elements of the matrix are $M_{ij} = t_{ij}(\boldsymbol{x})$, $i, j = 1, \dots, k$.

2.4 Computing trade-off rates by definition

With the help of trade-off rate information $t_{ij}(\boldsymbol{x}^*, \boldsymbol{d})$ at the Pareto optimal point $\boldsymbol{x}^*$ we can study what is the ratio of change involving the objective function f_i and f_j when we move to a certain direction $\boldsymbol{d}$. In other words, we can see how objective f_i is going to change if we, for example, want to improve the value of objective f_j. If we, for instance, select $\boldsymbol{d} = -\nabla f_j(\boldsymbol{x}^*)$ in formula (3) we can compute a total trade-off rate that reflects how the values of objectives f_i ($i = 1, \ldots, k$ and $i \neq j$) are locally going to change if we improve objective f_j.

Next, we define a regular point which characterizes when partial trade-off rates can be used.

Definition 2.5
A point $(\boldsymbol{x}^*, \delta^*)^T \in S \times \mathbb{R}$ *of problem* $(P(\bar{\boldsymbol{z}}))$ is regular *if the gradients of active inequality constraints, that is,* $\{(w_i \nabla f_i(\boldsymbol{x}^*), -1)^T \mid w_i(f_i(\boldsymbol{x}^*) - \bar{z}_i) = \delta,\ i = 1, \ldots, m\}$, *are* linearly independent.

Partial trade-off rate makes sense only in regular points. This can be very loosely speaking elaborated in such a way that if our decision space is two dimensional and we have three objectives, then all the Pareto optimal points are regular if gradients are in pairwise comparison linearly independent. If we add one more objective, then regularity is not in a general case fulfilled. This same extends to situations when the dimension n of the decision space is three or more. In general, if we have more than $n + 1$ active constraints, then there are no regular points because there can be only $n + 1$ linearly independent vectors in an $n + 1$ dimensional space. Partial trade-off rate indicates how much improvement in objective value f_j is going to degrade value of objective f_i, while all the other objectives f_l, $l \neq i, j$, remain at their current levels. In other words pairwise trading-off of two objectives is possible only when there exist such a direction where other objectives are not changing.

As proposed in [35], partial trade-off rate can be also computed from (3) using projection matrix $\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}\boldsymbol{J}$ where $\boldsymbol{J}$ is a Jacobian matrix of objectives f_l ($l \neq i, j$), that is, matrix which has vectors $\nabla f_l(\boldsymbol{x})^T$ as row vectors. If we now use direction $\boldsymbol{d} = \boldsymbol{P}(-\nabla f_j(\boldsymbol{x}))$, where $f_l(\boldsymbol{x})^T \boldsymbol{d} = 0$ for all $l \neq i, j$ we obtain a partial trade-off rate involving objectives f_i and f_j. In what follows, we will use this method to verify partial trade-offs obtained with the method of Sakawa and Yano.

2.5 Computing trade-off rates using KKT multipliers

Problem $(P(\bar{\boldsymbol{z}}))$ is a special case of the hyperplane scalarization method proposed by Sakawa and Yano [32, 40]. In [32], the authors presented a method to compute partial trade-off rates for the hyperplane scalarization with the help of Karush–Kuhn–Tucker (KKT) multipliers [14, 17]. In this paper, we utilize their approach in computing partial trade-off rates. Next, we give some theoretical results connecting the trade-off rates and the KKT multipliers of problem $(P(\bar{\boldsymbol{z}}))$. The presentation is based on [32].

The feasible reference point set Y for problem $(P(\bar{\boldsymbol{z}}))$ is defined by $Y = \{\bar{\boldsymbol{z}} \in \mathbb{R}^k \mid w_i(f_i(\boldsymbol{x}) - \bar{z}_i) \leq \delta,\ i = 1, \dots, k,\ \boldsymbol{x} \in \mathbb{R}^n, \delta \in \mathbb{R}\}$. Next, we give second order sufficient optimality conditions.

Theorem 2.6
If the functions defining problem $(P(\bar{\boldsymbol{z}}))$ are twice continuously differentiable in a neighborhood of $(\boldsymbol{x}^*, \delta^*)^T$, then $(\boldsymbol{x}^*, \delta^*)^T$ is a unique local optimal solution of problem $P(\bar{\boldsymbol{z}})$ if there exist (KKT multiplier) vector $\boldsymbol{\lambda}^* \in \mathbb{R}^k$ such that

$$\begin{aligned} w_i(f_i(\boldsymbol{x}^*) - \bar{z}_i) - \delta^* \leq 0, &\quad i = 1, \dots, k, \\ \lambda_i^*(w_i(f_i(\boldsymbol{x}^*) - \bar{z}_i) - \delta^*) = 0, &\quad i = 1, \dots, k, \\ \lambda_i^* \geq 0, &\quad i = 1, \dots, k, \\ \sum_{i=1}^k \lambda_i^* w_i \nabla f_i(\boldsymbol{x}^*) = 0 & \\ \sum_{i=1}^k \lambda_i^* = 1 & \end{aligned}$$

and, further, if

$$\begin{aligned} \boldsymbol{y}^T F(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \boldsymbol{y} > 0 &\quad \text{for all } \mathbf{0} \neq \boldsymbol{y} \in \mathbb{R}^{n+1} \text{ such that} \\ \bar{\nabla} f_i(\boldsymbol{x}^*)^T \boldsymbol{y} \geq 0 &\quad \text{for all } i, \text{ where } f_i(\boldsymbol{x}^*) = \delta^*/w_i + \bar{z}_i \text{ and} \\ \bar{\nabla} f_i(\boldsymbol{x}^*)^T \boldsymbol{y} = 0 &\quad \text{for all } i, \text{ where } \lambda_i^* > 0, \end{aligned}$$

and, where $F(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is the matrix

$$\begin{bmatrix} \sum_{i=1}^k \lambda_i^* \nabla^2 f_i(\boldsymbol{x}^*) & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix},$$

$\mathbf{0}$ is the vector with all n components equal to zero and $\bar{\nabla} f_i(\boldsymbol{x}^*) = (w_i \nabla f_i(\boldsymbol{x}^*), -1)^T$ is the gradient of f_i with respect to both $\boldsymbol{x}$ and δ.

Proof. Follows directly from the general second order sufficient optimality conditions (see, for example, [4]).

In the following, let us assume that $(\boldsymbol{x}^*, \delta^*)^T$ is a unique local optimal solution of problem $(P(\bar{\boldsymbol{z}}^*))$ where $\bar{\boldsymbol{z}}^*$ is the corresponding reference point. Let us make the following assumptions:

(i) $(\boldsymbol{x}^*, \delta^*)^T$ is a regular point of the constraints of problem $(P(\bar{\boldsymbol{z}}^*))$,

(ii) the second order sufficient conditions of Theorem 2.6 are satisfied at $(\boldsymbol{x}^*, \delta^*)^T$ and

(iii) there are no degenerate constraints, that is, if $w_i(f_i(\boldsymbol{x}) - \bar{z}_i) = \delta$, then $\lambda_i > 0$.

The connection between partial trade-off rates and the KKT multipliers of problem $(P(\bar{\boldsymbol{z}}))$ arise from the following sensitivity theorem which can be found, for example, in [4].

Theorem 2.7
Let $(\boldsymbol{x}^*, \delta^*)^T$ be a unique local optimal solution of problem $(P(\bar{\boldsymbol{z}}^*))$ satisfying assumptions (i)–(iii). Let $\boldsymbol{\lambda}^*$ denote the KKT multipliers corresponding to the constraints of problem $(P(\bar{\boldsymbol{z}}^*))$. Then, there exist continuously differentiable vector-valued functions $\boldsymbol{x}(\bar{\boldsymbol{z}})$, $\delta(\bar{\boldsymbol{z}})$ and $\boldsymbol{\lambda}(\bar{\boldsymbol{z}})$ defined on some neighborhood $N(\bar{\boldsymbol{z}}^*) \cap Y$ so that $\boldsymbol{x}(\bar{\boldsymbol{z}}^*) = \boldsymbol{x}^*$, $\delta(\bar{\boldsymbol{z}}^*) = \delta^*$ and $\boldsymbol{\lambda}(\bar{\boldsymbol{z}}^*) = \boldsymbol{\lambda}^*$, where $(\boldsymbol{x}(\bar{\boldsymbol{z}}), \delta(\bar{\boldsymbol{z}}))^T$ is a unique local optimal solution of the problem $(P(\bar{\boldsymbol{z}}))$ for any $\bar{\boldsymbol{z}} \in N(\bar{\boldsymbol{z}}^*) \cap Y$ satisfying (i)–(iii) and $\boldsymbol{\lambda}(\bar{\boldsymbol{z}})$ is the KKT-multiplier corresponding the constraints of $(P(\bar{\boldsymbol{z}}))$. In addition,

$$\frac{\partial \delta(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} = -\lambda_i(\bar{\boldsymbol{z}}), \quad i = 1, \ldots, k,$$

on some neighborhood $N(\bar{\boldsymbol{z}}^*) \cap Y$.

It can be shown [32] that there exists functions $\bar{f}_j(\bar{\boldsymbol{z}}) = f_j(\boldsymbol{x}(\bar{\boldsymbol{z}}))$ for all $j = 1, \ldots, k$. The functions $\bar{f}_j$ are continuously differentiable because of the same property of the functions f_j. Furthermore, it follows from the sensitivity theorem and the continuous differentiability of $\bar{f}_j$ that

$$\left.\frac{\partial \bar{f}_i(\bar{\boldsymbol{z}})}{\partial \bar{f}_j}\right|_{\bar{\boldsymbol{z}} \in N(\bar{\boldsymbol{z}}^*)} = -\frac{\lambda_j(\bar{\boldsymbol{z}})}{\lambda_i(\bar{\boldsymbol{z}})} \quad \text{for all } i, j = 1, \ldots, k.$$

Let $\boldsymbol{x}^*$ be a local Pareto optimal solution. It can be shown [32], that

$$t_{ij}(\boldsymbol{x}^*) = \frac{\partial f_i(\boldsymbol{x}^*)}{\partial f_j} = \frac{\partial \bar{f}_i(\bar{\boldsymbol{z}}^*)}{\partial \bar{f}_j} = -\frac{\lambda_j^*}{\lambda_i^*} \quad \text{for all } i, j = 1, \ldots, k.$$

Thus, under the assumptions (i)–(iii) the partial trade-off rates can be calculated from the KKT multipliers. Note that trade-offs do not explicitly depend on the aspiration levels but implicitly through the KKT multipliers.

Furthermore, by selecting a suitable direction $\boldsymbol{d}^*$ in (3), we can obtain a connection to the KKT multipliers: For the optimal aspiration levels $\bar{z}_i^*$, $i = 1, \ldots, k$, we get

$$1 = \frac{\partial \bar{f}_i(\bar{\boldsymbol{z}})}{\partial \bar{f}_i} = \frac{\partial \bar{f}_i(\bar{\boldsymbol{z}})}{\partial(\delta/w_i + \bar{z}_i)} = \frac{\partial \bar{f}_i(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} \tag{4}$$

$$= \frac{\partial \bar{f}_i(\bar{\boldsymbol{z}})}{\partial x_l}\frac{\partial x_l(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} = \frac{\partial f_i(\boldsymbol{x}(\bar{\boldsymbol{z}}))}{\partial x_l}\frac{\partial x_l(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} \overset{\bar{\boldsymbol{z}}=\bar{\boldsymbol{z}}^*}{=} \nabla f_i(\boldsymbol{x}^*)^T \boldsymbol{d}^* \tag{5}$$

and

$$-\frac{\lambda_i}{\lambda_j} = \frac{\partial \bar{f}_j(\bar{\boldsymbol{z}})}{\partial \bar{f}_i} = \frac{\partial \bar{f}_j(\bar{\boldsymbol{z}})}{\partial(\delta/w_i + \bar{z}_i)} = \frac{\partial \bar{f}_j(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} \tag{6}$$

$$= \frac{\partial \bar{f}_j(\bar{\boldsymbol{z}})}{\partial x_l}\frac{\partial x_l(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} = \frac{\partial f_j(\boldsymbol{x}(\bar{\boldsymbol{z}}))}{\partial x_l}\frac{\partial x_l(\bar{\boldsymbol{z}})}{\partial \bar{z}_i} \overset{\bar{\boldsymbol{z}}=\bar{\boldsymbol{z}}^*}{=} \nabla f_j(\boldsymbol{x}^*)^T \boldsymbol{d}^*, \tag{7}$$

where, $\boldsymbol{d}^* = (d_1^*, \ldots, d_k^*)^T$, $d_l^* = \partial x_l(\bar{\boldsymbol{z}}^*)/\partial \bar{z}_i$. Therefore,

$$-\frac{\lambda_j}{\lambda_i} = \frac{\nabla f_i(\boldsymbol{x}^*)^T \boldsymbol{d}^*}{\nabla f_j(\boldsymbol{x}^*)^T \boldsymbol{d}^*} = t_{ij}(\boldsymbol{x}^*, \boldsymbol{d}^*).$$

Let us point out that for the direction $\boldsymbol{d}^*$ it holds that $\nabla f_l(\boldsymbol{x}^*)^T \boldsymbol{d}^* = 0$ for all $l = 1, \ldots, k$ and $l \neq i, j$.

3 Using trade-off information to support the decision maker

Traditionally, trade-off information has been used as an essential part of some multiobjective optimization method in guiding the search towards the most preferred solution (ISWT [7], method of Halme and Korhonen for linear problems [28], STOM [31], ISTM [37], ZW [41]). Opposed to this, we use trade-off information as a supporting tool for the DM, not as a part of the method itself as mentioned previously. In other words, our aim is not to develop a new interactive method but to support using the existing ones without concentrating on some particular method. As an example, we use the achievement scalarizing function approach in order to illustrate the ideas presented as already mentioned. In a basic variation of achievement functions, the weights $w_i = 1/(z_i^{nad} - z_i^{**})$, $i = 1, \ldots, k$, are used [36]. Examples of other methods utilizing reference points are the reference point method by Wierzbicki [36], the visual interactive approach [15], the GUESS method [2] and the light beam search [11].

In this paper, we demonstrate our ideas with a reference point based method, but it is possible to use these ideas in classification based methods also. This is due the fact that there is a connection between a classification and a reference point. Namely, if the DM makes a classification in some Pareto optimal solution, a reference point can be produced [27]. As an example of classification based methods we can mention the NIMBUS® method and the satisficing trade-off method [29, 31].

In interactive multiobjective optimization, preference information requested from the DM is needed, for example, in specifying a reference point or classifying the objective functions in different classes. Our aim is to support the DM in the selection of the next reference point or making the next classification. Our hypothesis is, that if we are able to obtain valid trade-off information, it can help the DM in selecting the next reference point or making the next classification and, therefore, the number of iterations needed can be reduced. Thus, with the help of the additional information shown to the DM, the whole interactive solution procedure can be shortened which saves the time of the DM and reduces the number of Pareto optimal solutions needed to be generated. The latter reason can be especially advantageous in solving real-world multiobjective optimization problems where a generation of one Pareto optimal solution might be computationally expensive [8].

To compute trade-off rates we utilize the ideas of Sakawa and Yano where trade-off rates can be obtained from the KKT multipliers as shown in Section 2. Note that if we are using a suitable optimizer this does not require any additional computation because the KKT multipliers are already available, this is the case especially with some of widely used SQP-based optimizers [5].

In addition to obtaining supporting information, it is crucial how it is presented to the DM. We do not want to increase the cognitive burden set on the DM too much by giving him/her too much detailed or potentially misleading information. Therefore, we do not want to directly present any numerical trade-off information to the DM, but just indication of the magnitude of change and on which direction the change can be predicted to occur.

New reference point selection or classification can be made in several ways using trade-off rate information. Because we are offering only supporting information at some Pareto optimal solution, the DM is totally free to choose how to utilize this information. Because of this, it is more convenient to offer partial instead of total trade-off rate information. Total trade-off information is always related to some specified direction and that is why it might be difficult to grasp an idea what kind of trade-offs there exist in some other direction. By using partial trade-off rate information, it could be easier to consider trade-off rates to different directions as a linear combination of partial trade-off rates.

3.1 Presenting trade-off information to the DM

In the previous section, we gave definitions for trade-off rates and also presented a couple of methods to compute them. Now, we are going to present some ideas about how this kind of trade-off information can be used in interactive reference point or classification based methods. In what follows, we propose two different ways to visualize trade-off information. On the one hand, our aim is to decrease cognitive load of the DM by hiding possibly inaccurate numerical data. On the other hand, we also try to speed up the process where the DM considers how a new reference point should be set.

Let us consider a setting where the DM is using, for example, our simplified interactive reference point based method to solve some particular multiobjective optimization problem. At every iteration the DM must decide, whether to quit the interactive procedure and consider the last solution as the most preferred, or how to set a new reference point in such a way that it produces a more desirable solution. Let us point out that reference point based methods have been criticized for their lack of ability to support the DM in selecting a new reference point during the interactive solution procedure.

As an example, let us assume that the DM is solving a problem of three objective functions and at iteration h some Pareto optimal solution $\boldsymbol{x}^h$ is obtained, this solution is depicted in Figure 2, where gray bars are related to the objectives f_1, f_2, and f_3. The black dots indicate objective function values of objective vector $\boldsymbol{z}^h = (z_1, z_2, z_3)^T$ at the current Pareto optimal solution, and the circles stand for the aspiration levels in reference point $\bar{\boldsymbol{z}}^h = (\bar{z}_1, \bar{z}_2, \bar{z}_3)^T$ given by the DM. The current objective vector $\boldsymbol{z}^h$ is produced using the reference point $\bar{\boldsymbol{z}}^h$. The values of the ideal and the nadir objective vectors are shown at the bottom and top of the box, respectively.

At this point, we assume that the DM is not completely satisfied with the current

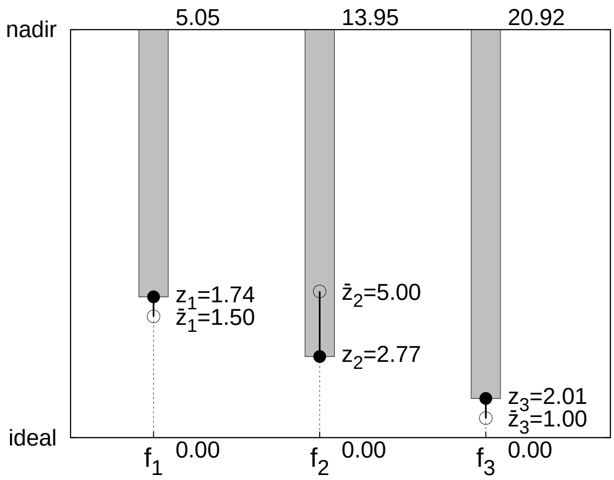

Figure 2: Objective function values at current solution

solution and wants to set a new reference point. Now, at least two possible questions can be considered *"How should the reference point be changed in order to achieve maximal gain in some objective(s) with minimal loss in others?"* or *"Is it possible to obtain a solution where a certain objective is improved by relaxing the aspiration level of some other objective while the rest of the objectives are staying close to their current values?"*

Reference point methods do not typically offer answers to the above kind of questions. Therefore, the only thing the DM can do is to vary the reference point around the current solution and use trial and error. Especially in the case of problems that have a higher number of objective functions, it might be very frustrating to probe around the objective space with a reference point.

Let us assume that in the current Pareto optimal solution presented in Figure 2, the DM is interested in slightly improving, for example, objective f_2 by sacrificing in objective values of f_1 or f_3. Here, we can support the DM by offering trade-off information related to objective values at the current solution. A straightforward approach to this is, of course, to show numerical trade-off rate information from the trade-off matrix. Let us assume that at the current solution we have a partial trade-off rate matrix available as presented in Table 1. From the partial trade-off rate matrix we see how the objective f_i at the column i is going to change if we want to improve the objective f_j at the row j. For example, if we want to improve objective f_2 by one unit in such a way that objective f_3 is to impair and objective f_1 is remaining fixed then the trade-off rate is -0.0209, where a negative value indicates increasing objective value.

Because Table 1 contains partial trade-off rates all the values are negative. The diagonal is empty because due to the Pareto optimality it is not possible to improve any objective without sacrificing in some other objective. Exception is the case where the considered point is weakly Pareto optimal, when some elements at diagonal might contain the value 1. In a total trade-off rate matrix all diagonal values are equal to 1 and off-diagonal values can be either negative or positive.

	f_1	f_2	f_3
f_1		-83.1412	-1.7341
f_2	-0.0120		-0.0209
f_3	-0.5767	-47.9445	

Table 1: Matrix of partial trade-off rates

In general case for nonlinear problems, the values in the trade-off rate matrix are often accurate approximations only in some finite neighborhood of the current solution. In this sense, it might be misleading to show these values directly to the DM because a proper neighborhood is problem dependent and it is difficult to characterize. Due to the local nature of trade-off rate information, there is a close connection to sensitivity theory which considers the effect of small perturbations in the problem data to the solutions obtained (see, for example, [4]). A direct analysis of numerical trade-off rate matrices might cause too much cognitive burden for the DM, especially, if this kind of analysis is carried out for several Pareto optimal solutions. Therefore, we now consider another way to carry out this analysis.

The numerical trade-off rates in Table 1 might be confusing to the DM. For example, if we consider one unit improvement in objective f_1 and are willing to impair objective f_2, according to the first row in Table 1, objective f_2 is going to impair approximately 83 units. However this does not make much sense because by looking at Figure 2 we see that objective f_2 gets values in the Pareto optimal set within the range $[0, 13.95]$. This leads to an idea that maybe it is enough to know that change will be in some sense "large enough".

One way to approach the above problem is to reduce accuracy of numerical trade-off rate information. Sometimes, it might be enough for the DM just to know whether the trade-off involving some objectives is below, equal, or above neutral rate of change. In such a case, a so-called *arrow matrix visualization* can be used. The arrow matrix in Figure 3 corresponds the partial trade-off rate matrix in Table 1. In this figure, the downward pointing triangles indicate a negative trade-off rate. Different colors define the magnitude of the trade-off rate; a white triangle denotes a small change, a gray triangle denotes a neutral change while a black triangle stands for a significant change. In this paper, we classify trade-off rates in such a way where the range $(-1/2, 0]$ correspond to a small change, the range $[-2, -1/2]$ stands for a neutral change and the range $(-\infty, -2)$ means a significant change. Note that by changing the limits $-1/2$ and -2, the division of the values of trade-off rates can be altered. An upward pointing triangle can be elaborated in the same way as a downward pointing one but it indicates positive trade-off rate values. Positive trade-off values are possible if we are inspecting total trade-off rates.

Most of the time the information level of the arrow matrix might be enough, but it is not always so helpful. If, for example, the DM wants to improve objective f_2 by relaxing either objective f_1 or f_3 but wants to choose the one which has better trade-off rate, then according to the arrow matrix in Figure 3 it is impossible to say which objective should

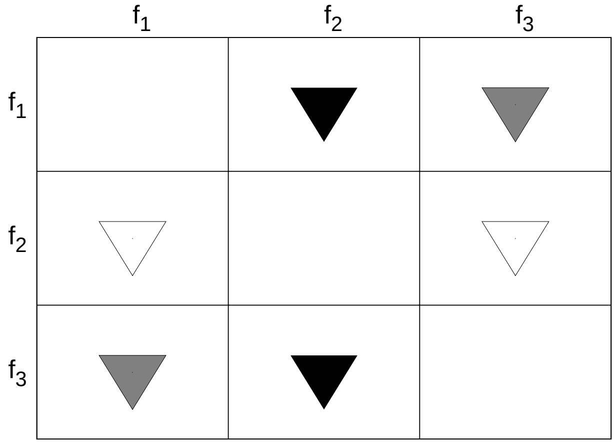

Figure 3: Arrow matrix visualization

be chosen. However, if both trade-off values are big enough it does not necessarily matter what the DM chooses. The arrow matrix does not reveal these kind of relationships.

As already mentioned, if trade-off rates are large enough in practice, it does not necessarily make any difference which objective really has the absolutely largest trade-off rate. Instead, it is more fruitful to know which trade-offs are “large enough” and which are close to a neutral trade-off rate. This leads to an idea where large trade-off rates are compressed to the same class indicating large enough change. One possible way to do this compression is to use a sigmoid function $s : \mathbb{R} \to (-1, 1)$,

$$s(t) = \tanh\left(\frac{t}{2}\right) = \frac{e^{t/2} - e^{-t/2}}{e^{t/2} + e^{-t/2}}. \tag{8}$$

In Figure 4 we can see trade-off visualization where trade-off values are taken from Table 1. Note that in all three boxes in the figure, the scale in the vertical axis is compressed with the sigmoid function. Every box in this Figure is related to one row of trade-off matrix. In the case of partial trade-off information, for example, the leftmost box is, on the one hand, reflecting how objective f_2 is going to change if we improve objective f_1 while f_3 remains fixed. On the other hand, the leftmost box also reflects how objective f_3 is going to change if we improve objective f_1 while f_2 remains fixed. According to this partial trade-off rate visualization we grasp an idea of how f_2 and f_3 are going to change if we just improve objective f_1 and let f_2 and f_3 change freely.

If in Figure 4 the black dot denoting the value of the trade-off rate for corresponding function is near the bottom of the box, then the trade-of rate is significant, in our classification this means that trade-off rate is less than -2. On the other hand, if the black dot is near the zero line, then the corresponding trade-off rate is small. In between, the trade-off rate can be considered neutral, which means that it is ”close” to one. Due to the presented classification of trade-off rates there is a connection between the compressed trade-off rate visualization in Figure 4 and the arrow matrix visualization in Figure 3.

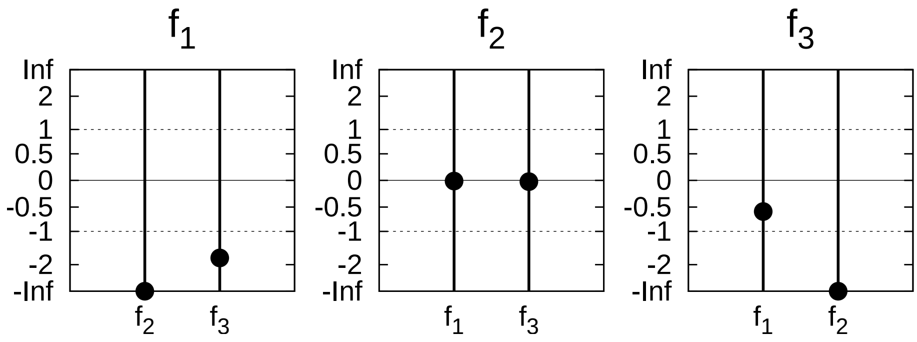

Figure 4: Visualization of compressed trade-off rates

If information in the arrow matrix or the compressed trade-off visualization is not accurate enough, the DM is of course always able to examine the numerical trade-off rate matrix directly. The main idea with these different visualizations is to reduce the cognitive burden of the DM by hiding the numerical information which can be inaccurate. Let us point out that it might be convenient to show the DM only the trade-off rate information when requested. Furthermore we can show, for instance, only one row at the time if the DM is only interested in improving some specific objective. By this way it is possible to reduce cognitive burden set on the DM even more. Finally, let us mention that utilizing trade-off information becomes more advantageous near the area where the most preferred solution could be obtained, in other words, when the DM has already found a promising solution and wants to study whether it is possible still to obtain some improvement without sacrificing too much in objective values at current solution.

4 Numerical example

In this section, we first demonstrate manually how the ideas presented in the previous sections can be utilized. After that, we present a simple automated test framework which we use to simulate the behavior of the DM, and try to reflect what kind of results can be obtained if our methods for visualizing trade-offs are used to support the DM.

We assume that the DM is using the presented simplified reference point method and during the interactive solution procedure is willing to see trade-off information to aid in how the next reference point should be placed. At this point, we have a few questions that we aim to answer: How useful is trade-off information in solving multiobjective optimization problems with interactive reference point based method, is it convenient to use the visualizations presented instead of numerical trade-off information and can we obtain reliable trade-off rates with the approach of Sakawa and Yano in practice? In all our tests, we use the following very simple academic problem

$$\begin{array}{ll} \text{minimize} & f_1(\boldsymbol{x}) = x_1^2 + x_2^2, \\ & f_2(\boldsymbol{x}) = 2(x_1-1)^2 + (x_2-2)^2, \\ & f_3(\boldsymbol{x}) = 3(x_1-2)^2 + (x_2-1)^2 \\ \text{subject to} & \boldsymbol{x} \in \mathbb{R}^2. \end{array} \tag{9}$$

For this problem, the ideal objective vector is $\boldsymbol{z}^* = (0.0, 0.0, 0.0)^T$ and the approximation for the nadir objective vector used is $\boldsymbol{z}^{\text{nad}} = (5.0512, 13.9499, 20.9186)^T$. In this case, the approximation of the nadir objective vector is known before hand and it is assumed to be accurate enough.

4.1 About implementation

In all our tests, we have used the SQP optimizer (E04UCF) of the NAG library. The SQP optimizer was used in such a way that no initial KKT multipliers were set. At the final solution, the KKT multipliers provided by the SQP optimizer were used to calculate the partial trade-off rate matrix with the method of Sakawa and Yano. At every solution, the partial trade-off matrix was also calculated directly from the definition by using the projection matrix, where objective function gradients computed at the last iteration of the SQP optimizer were used. In computing the projection matrix, the inverse matrix was computed by using the DPOTRI procedure from the LAPACK library. All the computational entities were implemented as independent programs by using a Fortran compiler in the HP-UX environment.

In the general case, the SQP optimizer is able to produce only locally optimal solutions for problem $(P(\bar{\boldsymbol{z}}))$, but because in this paper we have only used convex problems we do not need to consider how to offer proper initial point for the SQP. Of course the visualization ideas presented are working with nonconvex problems also, but in such a case, some global optimizer could be needed in order to offer initial solutions to the SQP optimizer. However, if we already have some global Pareto optimal solution for nonconvex problem and are interested only in local improvements, then the SQP optimizer is enough, and no computationally expensive global optimizers are needed.

4.2 Manual example

The following case study demonstrates how the presented trade-off rate visualization methods can be used to support the DM in practice. Let us now take the role of the DM and use the simplified interactive reference point method presented in Section 2 to solve problem (9).

First, we define the aspiration levels $\bar{z}_i \in [z_i^*, z_i^{\text{nad}}]$ for individual objectives f_i, $i = 1, \ldots, k$, constituting a reference point $\bar{\boldsymbol{z}}$. Let the initial reference point be, for example, $\bar{\boldsymbol{z}}^1 = (1.5, 5.0, 1.0)^T$. By solving problem $(P(\bar{\boldsymbol{z}}^1))$, we obtain a solution $\boldsymbol{z}^1 = (1.7433, 2.7710, 2.0073)^T$ which is visualized in Figure 2.

Let us now suppose that we want to improve objective f_2. According to the arrow matrix visualization in Figure 3, improvement in objective f_2 should be possible without affecting too much the current values of the objectives f_1 and f_3. In Figure 4, we can see even more clearly that trade-off structure for objective f_2 is very promising.

Now, we define a new reference point where the aspiration level for objective f_2 is improved by 1% in interval $[z_2^*, z_2^{\mathrm{nad}}]$ while the other aspiration levels remain at the same level with the objective vector $\boldsymbol{z}^1$ related to the current solution. In other words, our new reference point is $\bar{\boldsymbol{z}}^2 = (1.7433, 2.6315, 2.0073)^T$. Solving problem $(P(\bar{\boldsymbol{z}}^2))$ produces a Pareto optimal objective vector $\boldsymbol{z}^2 = (1.7446, 2.6351, 2.0126)^T$. We can compare objective vector $\boldsymbol{z}^1$ to $\boldsymbol{z}^2$ through the difference $\boldsymbol{z}^1 - \boldsymbol{z}^2$ and if in addition each component of this difference vector is scaled with the appropriate objective range $(z_i^{nad} - z_i^*)$ we get $(-0.0003, 0.0097, -0, 0003)^T$. In other words, an improvement of 1% in objective f_2 was achieved without affecting the other objectives too much. The example above demonstrates a situation where the DM, for instance, has already found some promising solution and wants to explore its surroundings without moving too far. Let us point out that this example is only reflecting our idea and we do not expect that the DM is going to make exactly 1% improvements. In real situation, the reference point can be of course set more freely.

4.3 Automated tests

We have used a simple automated test setting to study what kind of results can be obtained when the selection of a new reference point is based on our arrow matrix visualization. The test was made with problem (9) and, at this point, our aim is only to give an idea of how the visualizations of trade-offs presented can be used in order to define a new reference point. There is not only one way to set a new reference point according to trade-off information. In our tests, we have adopted very simple approach which simulates the one we used in the manual example.

In the test setting, we generated 1000 random reference points $\bar{z}_i^1 \in [z_i^*, z_i^{\mathrm{nad}}]$, $i = 1, \ldots, k$, for problem (9). For each reference point, problem $(P(\bar{\boldsymbol{z}}^1))$ was solved and the partial trade-off rate matrices M_{SY} and M_P were computed by using the method of Sakawa and Yano and the projection method, respectively, for the solutions obtained. A new reference point was chosen by using the matrix M_{SY} and the matrix M_P is only used to verify computations. The matrix M_{SY} was used in such a way that the objective f_j were chosen to be improved if the corresponding trade-off rates indicated only small change for the other objectives. If some objective had this kind of promising trade-off rate structure at the current solution, then a new reference point was generated from the objective vector $\boldsymbol{z}^1$ related to the current solution by improving the aspiration level of objective f_j by 1% in the interval $[z_j^*, z_j^{\mathrm{nad}}]$. In other words, the improvement of the aspiration level for the objective f_j by 1% means that a new aspiration level $\bar{z}_j{}^2 = z_j^1 - 0.01(z_j^{\mathrm{nad}} - z_j^*)$ is chosen, while the values of $\boldsymbol{z}^1$ are chosen as the other aspiration levels, that is, $\bar{z}_i{}^2 = z_i^1$, where $i = 1, \ldots, k$ and $i \neq j$. Let us point out that 889 solutions out of the 1000 generated fulfilled the presented condition of promising trade-off structure. The remaining 111 points were not considered at all in this test.

Next, problem $(P(\bar{\boldsymbol{z}}^2))$ was solved by using the constructed reference point. To measure the goodness of the new solution $\boldsymbol{z}^2$ obtained we computed a normed difference $(z_j^1 - z_j^2)/(z_j^{\text{nad}} - z_j^*)$ for objective f_j selected to be improved. The normed average $\left(\sum_{i=1,i\neq j}^{k}(z_i^1 - z_i^2)/(z_i^{\text{nad}} - z_i^*)\right)/(k-1)$ was computed to reflect the average change in the other objectives. In Figure 5, the black plot depicts the normed difference in the improved objective f_j over all 889 points, and the gray plot depicts the normed average difference in the other objectives. It can be seen, that for all the generated reference points, the improvement in the selected objective f_j was quite close to 1%, while the average was about 0.096. For the other objectives, the normed average difference was quite close to zero.

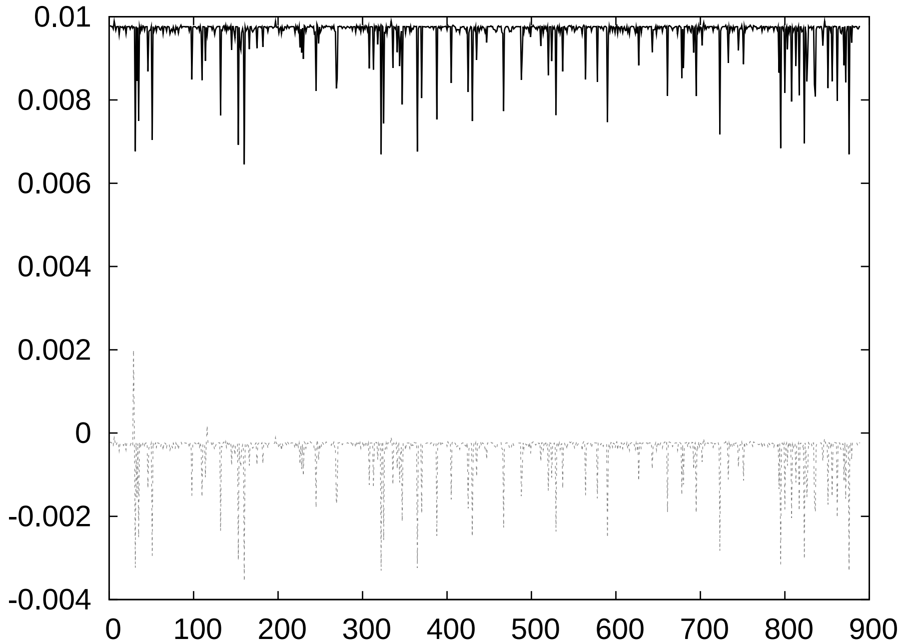

Figure 5: Normed changes using 1% step

An improvement of 1% is of course more or less arbitrary. For highly nonlinear problems the improvement should be quite small because trade-off rate is only a first order approximation. Namely, if we increase the amount of improvement, for example, to 5%, then results for our simple test problem start to behave in a more unpredictable way. This kind of a behaviour can be clearly seen in Figure 6. The average of normed difference in the improved objective f_j is in this case better than the one with 1% improvement (in Figure 5), but we must underline that our test problem is very simple and, in general case, an improvement of 5% might be way too large and the results can be almost anything.

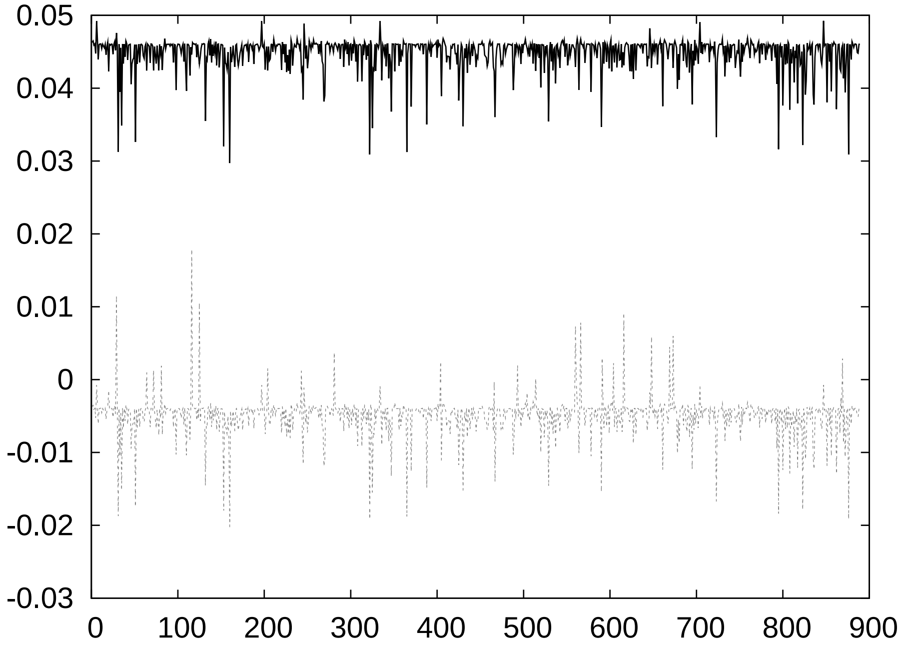

Figure 6: Normed change using 5% step

4.4 On computational reliability of trade-off rates

Although the partial trade-off rate matrix $M(\boldsymbol{x})_{SY}$ was used in all the computations, in our tests we always computed also the matrix $M(\boldsymbol{x})_P$ to verify numerical reliability. The matrix $M(\boldsymbol{x})_P$ was computed using the objective function gradients computed at the last iteration of the SQP optimizer. The difference of these two matrices was computed and the standard Euclidean matrix norm $||A||_2 = \sqrt{\sum \text{diag}(A^T A)}$ of the difference was used to reflect the total error between these two matrices. In other words, at every Pareto optimal solution $\boldsymbol{x}$ the total error between the trade-off matrices $M(\boldsymbol{x})_{SY}$ and $M(\boldsymbol{x})_P$ was characterized with the norm $||M(\boldsymbol{x})_{SY} - M(\boldsymbol{x})_P||_2$.

In Figure 7, this error measure is depicted for every solution computed in the test. In Table 2, maximum, average, median, and minimum of the pairwise difference between the components of the trade-off matrices obtained from $|M_{SY,ij} - M_{P,ij}|$ are reported. These results indicate that most of the time the matrices $M(\boldsymbol{x})_{SY}$ and $M(\boldsymbol{x})_P$ are equal and if there is an error it is quite small. A closer examination of individual errors reveals that in all the cases, the difference seems to be related to the accuracy of the KKT multipliers produced by the SQP optimizer. Again, because our test problem is so simple no conclusions can be made for a general behavior, but at least these results indicate that, in this case, both of the methods are able to compute partial trade-offs similarly.

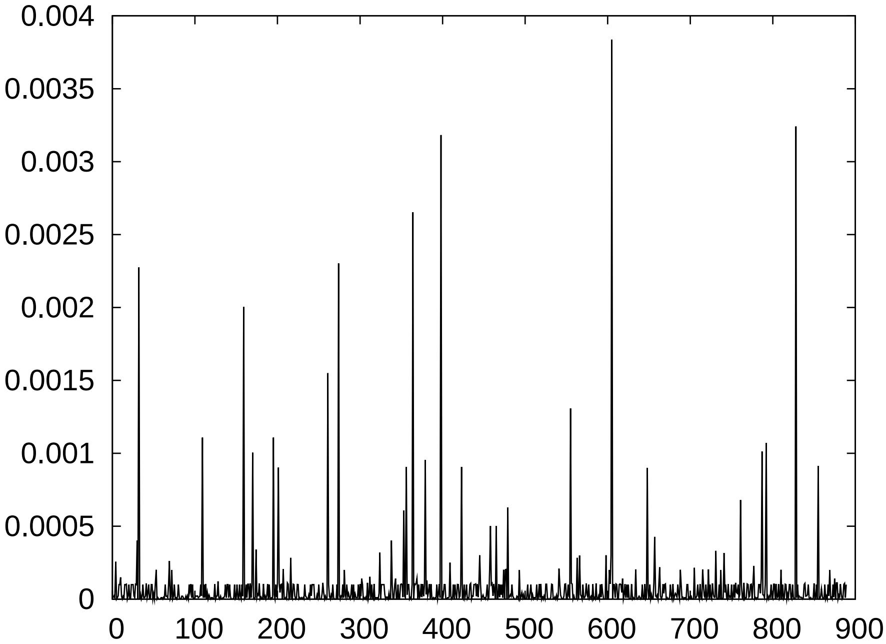

Figure 7: Error between trade-off rate matrices

Maximum	Average	Median	Minimum
0.023000	0.000040	0.000000	0.000000

Table 2: Trade-off matrix error analysis

5 Conclusions

We have presented some ideas for utilizing trade-off information in supporting the DM in finding better solutions for the multiobjective optimization problem in question. More precisely, we offer additional information in order to help the DM in selecting the next reference point or making the next classification. Our aim has been to present trade-off information directly to the DM and not to use it as a part of the interactive method itself as has been done previously. In other words, we have not developed a new interactive method, but supporting information can be used in connection with different interactive methods already available. We have used partial trade-off rates to predict the changes of the values of the objective functions.

Another target has been to study how trade-off information could be obtained without any additional computational effort. The method of Sakawa and Yano enables computation of partial trade-off rates from the Karush–Kuhn–Tucker multipliers without any additional computation. We are also interested in practical reliability of trade-off information so that it is worthwhile to present it to the DM.

Finally, we introduced two ways of visualizing the trade-off information. In our approach, no numerical information is shown to the DM because possibly inaccurate information can increase the cognitive burden set on to the DM and it can be difficult to interpret. Instead, we show the DM the predicted directions of the changes of the functions and the magnitude of the change. Related to these visualizations, we introduced a categorization of the amount of change predicted by trade-off information. Three different categories were identified, namely small, neutral and significant change.

We have studied these preliminary issues through a test problem and a set of reference points. The preliminary results showed that reliable trade-off information can be obtained for our problem. In addition, the results show that it can be useful in selecting the next reference point during the interactive solution procedure for our problem. We found that the reference points selected based on trade-off information and our visualizations of trade-off information provided solutions where more was gained than lost. Therefore, with this approach the most preferred solution can be found more quickly because the DM has more information available.

6 Future directions

This research has been a promising start in studying supporting the DM with trade-off information. Several ideas for future research have also been obtained. First of all, we need to perform tests with more problems. Here, we considered only unconstrained problems, so naturally we are interested in whether our approach is valid also for constrained problems.

Furthermore, in this paper we have only considered partial trade-off rates, but as mentioned, depending on the problem structure it is not always possible to compute them. In this sense, the total trade-off rates should be also considered. Especially if the DM is able to specify some specific direction of interest, then total trade-off rates could be more useful. The selection of proper direction is related to the way how the DM wants to set a new reference point or make a new classification.

In addition, we would be interested in providing such a trade-off information to the DM that could be useful also a little further from the current solution than the trade-off rates based on gradient information. This leads us to study how we can possibly reflect the reliability of linear trade-off approximation, in other words, in what kind of neighborhood of current solution the approximation is reliable enough. Finally, when we have made more tests and developed our ideas are further, some real-world problems could be solved with real DMs involved.

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