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HELSINGIN KAUPPAKORKEAKOULU
HELSINKI SCHOOL OF ECONOMICS
PL 1210
FI-00101 HELSINKI
FINLAND

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Incentive Option Valuation under Imperfect Market and Risky Private Endowment

Markku Kallio and Antti Pirjetä
Helsinki School of Economics
P.O. Box 1210, FIN–00101 Helsinki
e-mail: kallio@hse.fi, pirjeta@hse.fi

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Abstract

We investigate valuation of incentive stock options in a realistic setting that features hedging restrictions and other market imperfections, such as transaction costs, interest rate spreads between borrowing and lending, and costs for short positions. We add labor income and calculate the impact of correlated income and investment risks. Both European and American options are considered. We develop supply curves, where option values depend on the sold amount. Without friction costs and labor income, this specialized model is the discrete-time equivalent of the Ingersoll (2006) executive option pricing model. We find that friction costs and labor income have a material impact on subjective option values.

Keywords: Executive compensation, incentives, option valuation
JEL classification: G11, G13, G32

1 Introduction

Valuation of incentive stock options (or executive stock options, ESOs) has received much attention in finance and microeconomics literature. This topic has both theoretical and practical importance given the major role of options in compensation packages. Practical importance follows from the fact that options generate a significant share of executive remuneration. According to a survey on CEO remuneration in the US, published by HR consulting firm Mercer, 265 out of 350 CEOs received stock options and they generated 52 percent of the value of long-run incentives [1], which includes common and restricted stock as well as stock options. Note that restricted stock grants can be viewed as incentive stock options with zero strike price.

The theoretical approach to managerial compensation aims to find an optimal contract that solves the principal-agent problem, in which the risk-neutral principal (body of shareholders) imposes incentives to a risk-averse manager. In general, it is difficult to characterize the optimal contract that implements the principal's desired action at least cost. By optimal we mean the 'second-best' contract referring to a setup, where the principal cannot fully monitor the agent. In their classical paper, Grossman and Hart [8] derive necessary and sufficient conditions for an increasing incentive scheme, a primary requirement for any incentive scheme. Aseff and Santos [1] show that under some limiting conditions, the optimal contract reduces to a package of fixed salary and stock options. This is the case, if one assumes log utility and the Grossman-Hart conditions for an increasing incentive scheme.

We stress that a single incentive item, such as an option, should not be valued in isolation of other items. In particular, different forms of labor income like fixed salary and bonuses should be considered when valuing incentive options. Below we demonstrate that inclusion of labor income in a consumption-investment problem has a large effect on incentive option values.

Based on valuation by indifference proposed by Pratt [25], the valuation problem of ESOs is usually solved by calculating the certainty equivalent of option cash flows in a discrete-time framework. Such subjective value is the manager's ask price; i.e., the minimum price at which the manager is willing to sell an option. There is a number of papers implementing ask price valuation in

[1] The source is Mercer Human Resource Consulting 2005 CEO Compensation Survey. An interesting result of this survey is that other equity-based compensation, such as restricted shares, have increased their share relative to stock options.

discrete time. For instance, Detemple and Sundaresan [6] calculate incentive option values for the manager who cannot sell short the underlying asset. In their model, if the constrained manager has concave utility, the certainty equivalent value is bounded above by the risk-neutral value. Detemple and Sundaresan develop a pricing kernel, where the short-sales constraint results in an implicit dividend yield reducing the ESO value. Hall and Murphy [9] give a detailed review of incentive option pricing problems and literature, and present a basic ask price model that values the option using a lognormal distribution for the terminal stock price. However, this model is conditional on the assumption that the CAPM holds.

Investigating early exercise has been important in the valuation discussion. Carpenter [4] presents a binomial model for American options with an exogenous probability for early exercise. Bettis et al. [2] explain early exercise by fitting regression models in an exercise data from the US. Nevertheless, the discrete-time models cited above do not feature endogenous exercise decision, with the exception of Detemple and Sundaresan [6].

Theoretical papers have dominated the incentive option literature given the scarcity of price data. However, there is a smaller market, the Helsinki Stock Exchange, where incentive options have been traded since 1999. The availability of ESO price data as well as simultaneous price data on the underlying stocks enables practical calibration of option pricing models. Based on data of 15,769 trades of ESOs issued by seven listed firms, Ikäheimo, Kuosa and Puttonen [12] calibrate a basic Black-Scholes model to ESO prices using historical volatility and find some evidence of underpricing.

There are also some studies solving the managerial portfolio problem and valuing incentive options using continuous time models. Typically these papers solve some extended version of the Merton portfolio problem, Merton [22] and [23]. Generally speaking, the advantage of continuous-time models, when applicable, is that they lead to closed-form solutions, and in many cases these solutions are intuitive, for example, allowing inference on how the correlation of income and investment risks affects portfolio choice. For example, Henderson [11] solves an extended Merton portfolio problem with a random income stream in an exponential utility framework. The choice of utility function allows her to derive closed-form solutions for portfolio weights as well as a number of useful results regarding the manager's hedging demand. While exponential utility brings the advantage that closed-form solutions exist for portfolio weights and certainty equivalent, the disadvantage is a constant absolute risk aversion

(CARA). CARA implies that if the manager's labor income is certain or the labor income risk is idiosyncratic[2], the resulting portfolio of risky assets is independent of labor income[3]. The continuous time model of Koo [18] solves the portfolio problem using power utility. Unfortunately, the model yields closed-form solutions only in a complete market case where income risk can be fully hedged in the financial market.

Power utility functions imply constant relative risk aversion. In this case, the level of income impacts portfolio choice, even if income and investment risks are uncorrelated; see Campbell and Viceira [3]. Ingersoll [13] uses power utility and develops an extension of the Black-Scholes model for the setup, where a minimum weight for employer's stock is required in the manager's portfolio. Ingersoll derives a subjective pricing kernel, where the risk-free rate is adjusted to accommodate for the portfolio constraint, and as a result the subjective value of the option will be less than its objective value. Note that the Ingersoll [13] solution is qualitatively similar to Detemple and Sundaresan [6]; in both papers the effect of portfolio constraints is to reduce the (subjective) risk-free rate, effectively decreasing the option value.

We adopt logarithmic and power utility for ask price valuation. For numerical analysis we take the Ingersoll model as a starting point and extend it in a discrete-time framework in several ways: we consider market imperfections, such as transaction costs, interest rate spreads between borrowing and lending, and costs for short positions. We also account for labor income and assume it may be correlated with risky asset returns; and we carry out option valuations assuming the manager considers holding some options. Davis, Kubler and Willen [5] consider a lifecycle model with risky investments and labor income, but without any options. They find that adding a borrowing-lending spread has a significant impact on optimal portfolios.

Our numerical analysis demonstrates the importance of taking market imperfections into account in subjective valuation of incentive options. Ingersoll [13] already points out the valuation impact, when the manager is forced to hold some company stock. We contribute by showing that friction costs (transaction costs, bid ask spreads, shorting costs, etc.) can have a significant impact in subjective valuation. Furthermore, as is well known in arbitrage pricing theory, in the perfect and complete market case, the valuation results are unique.

[2] there is no co-variation in labor income and stock return.

[3] For discussion of disadvantages of CARA utility, see pp. 166–167 of Campbell and Viceira [3].

However, even mild friction costs result in a relatively wide arbitrage-free price interval.

Neither Detemple and Sundaresan [6] nor Ingersoll [13] consider friction costs at all. Furthermore, their approach do not apply to cases where manager's salary is included in the analysis. Yet, we demonstrate that the relative impact of salary in a subjective option value can be tens of percents. Similarly, if the manager considers selling some of the options in possession, then exercising the rest yield a stochastic cash flow stream, which has an impact in subjective valuation of the options to be sold. Such valuation does not fall within the framework of Ingersoll either; yet, we show that the valuation impact can be quite dramatic.

The rest of the article is organized as follows. In Section 2, we introduce a methodology for marginal ask price valuation. A discrete time double binary tree setup is employed to support numerical analysis in subsequent sections. Without friction costs and labor income, the specialized model is a discrete time equivalent of Ingersoll [13]. A general theory in a discrete setting together with proofs is presented in the Appendix and the double binary tree setup is a simplified specialization of the general theory. Such a choice was made in hope to improve readability of the article.

Section 3 concentrates on a variety of European and American options with and without friction costs, dividends and a vesting period. Accounting for market frictions results in bounds for the arbitrage free option value. In some instances the ask price of the manager is below the lower bound implying that the manager is willing to sell at any arbitrage free price. In other cases, the ask price is strictly within the bounds so that, depending on the arbitrage free price, the manager may or may not be willing to sell. Exercising American options is endogenous in the model. Due to market frictions, the results show that early exercise may be optimal even without dividends.

Section 4 develops the ask price function for an option; i.e., the subjective ask price as a function of fraction sold. As in standard microeconomic theory, the intersection of the price function and the (equilibrium) market price determines the number of options which the manager is willing to sell at the market price. As expected, the price function in strictly increasing. Numerical results indicate that the price can be highly sensitive with respect to the fraction sold.

In Section 5 we find that labor income has a significant effect in the subjective valuation of incentive options. If the labor income risk is deterministic or idiosyncratic, subjective value of options will in fact increase. However, if the labor income and investment income risks are correlated, the effect can be either positive or negative.

In Section 6 we report two case studies: one for Fortum (the second largest electric utility firm in the Nordic countries) and another for Nokia (the global market leader in mobile phones). ESOs of both firms are traded in the Helsinki Stock Market. Based on interviews with executives of these companies, several observations of interest arise. For instance, due to insider restrictions, trading only can take place after quarterly reports. Hence, the use of a discrete-time model with a three month period is supported. Also subjective views on the future success of company's strategy play a more important role than the views present in the stock market. Hence, the subjective value of an ESO can deviate substantially from the market price.

2 Subjective valuation of an incentive option

In this section, we introduce marginal indifference pricing for option valuation and discuss data employed for numerical analysis in subsequent sections. The aim is to determine the ask price; i.e., the price at which the manager is indifferent between selling and not selling an option. While the method originates from the certainty equivalent concept by Pratt (1964), the resulting option value is consistent with arbitrage theory. Numerical evaluation of the marginal ask price is based on a consumption-investment model with expected utility maximization. The plain problem of valuing ESOs has been solved by many. We contribute to this discussion by introducing an enhanced valuation method. It accounts for realistic market imperfections - such as transaction costs, interest rate margins and short position costs - and private endowments, such as a risky salary. Due to complexity of the problem, computations are carried out in a discrete setting employing stochastic programming.

2.1 A consumption-investment model

The continuous time model of Ingersoll [13] involves three assets: a risk-free asset, the market (index) portfolio and the stock of the employer. The two risky assets follow a bivariate GBM. We proceed by employing a discretized version of this model, where the discrete time process converges to the bivariate GBM as the time steps approach zero. We also consider an exogenous stochastic cash flow stream, a private endowment process accounting for risky labor income, for instance. Hence, unlike in Ingersoll [13], such cash flow can be nonzero after the initial stage.

r	0.05	logarithm of the (total) risk free return
ν_m	0.04	logarithmic (total) increment of the index
q_m	0	dividend yield of the index
σ_m	0.4	volatility of the index
ν	$r + \beta(\nu_m - r)$	logarithmic (total) increment of the stock price
q	0.01	dividend yield of the stock
σ	0.3	volatility of the stock
v	0.2	idiosyncratic volatility of the stock
β	$[(\sigma^2 - v^2)/\sigma_m^2]^{0.5}$	beta of the stock with respect to index

Table 1: Single period parameters and annual data for the discrete model.

In discrete time setting the time span of N years is subdivided into T periods of $\Delta = N/T$ years with stages $t = 0, 1, \ldots, T$. The logarithm of the total risk free return in each period is a constant r. The return of the index and the stock price are stochastic and interdependent. With dividend yield q_m, expected cum dividend value ν_m and variance σ_m^2, the logarithm of the index increases in each period by $\nu_m - q_m + \tilde{u}$, where $\tilde{u}$ is a stochastic increment with $\mathrm{var}(\tilde{u}) = \sigma_m^2$. Employing a stochastic increment $\tilde{v}$ with $\mathrm{var}(\tilde{v}) = v^2$, the logarithmic increment of the stock price is $\nu - q + \beta\tilde{u} + \tilde{v}$ with dividend yield q, drift ν and variance σ^2. Assuming that $\tilde{v}$ and $\tilde{u}$ are independent, $\sigma^2 = \beta^2\sigma_m^2 + v^2$. The notation together with data employed in Sections 3–5 is summarized in Table 1.

Stochastic processes of the stock price, index and exogenous cash flow are approximated by a double binary event tree. It reveals realizations of prices and exogenous cash flows. The nodes of the tree are denoted by $k = 0, 1, 2, \ldots$, with a root node $k = 0$ at time $t = 0$. Node probabilities are $\pi_k > 0$. As illustrated in Figure 1, at each stage $t < T$, given a node k associated with a level of the index and a stock price, there are four successor nodes j at stage $t+1$. For the index, there are two realizations σ_m and $-\sigma_m$ for $\tilde{u}$. Hence, there are two realizations for logarithmic increment $\nu_m - q_m + \tilde{u}$ of the index. Also for the stock, there are two realizations v and $-v$ for $\tilde{v}$. Hence, there are four realizations of increments $\nu - q + \beta\tilde{u} + \tilde{v}$ in nodes j. With an equal probability for each node j, one can readily check that our choice matches the expected values of $\nu_m - q_m$ and $\nu - q$, the variances σ_m^2 and v^2, and the covariance relation $\sigma^2 = \beta^2\sigma_m^2 + v^2$ holds.

			logarithmic increments	
k	j	p_j	index	stock
	•	1/4	$\nu_m - q_m + \sigma_m$	$\nu - q + \beta\sigma_m + v$
	•	1/4	$\nu_m - q_m + \sigma_m$	$\nu - q + \beta\sigma_m - v$
•				
	•	1/4	$\nu_m - q_m - \sigma_m$	$\nu - q - \beta\sigma_m + v$
	•	1/4	$\nu_m - q_m - \sigma_m$	$\nu - q - \beta\sigma_m - v$

Figure 1. Part of the event tree: Node k and its four successor nodes j with logarithmic increments and conditional probabilities $p_j = \pi_j/\pi_k = 1/4$.

Endogenous variables of the model are defined for each node k as follows. For non-terminal nodes k, c_k denotes consumption in the period starting at node k. For terminal nodes, c_k is the total value of terminal positions. For all nodes k, asset positions taken at k are endogenous. Initial positions while entering time $t = 0$ are fixed. At each non-terminal node k, positions change due to purchases and sales. At terminal nodes no trading takes place.

Position dynamics equations for each asset are defined by node. We also consider subjective portfolio restrictions. Such restrictions may set bounds on portfolio positions, for instance. To conform to Ingersoll [13], we require that the weight on stock in manager's investment portfolio is at least $\alpha \geq 0$.

For all k, let e_k denote a private exogenous endowment of the manager. Then for node k, the cash balance equation is given by in- and out-flows resulting from a number of sources: the level of consumption c_k is equal to the exogenous cash flow e_k incremented by the cash flow from changes in asset positions, dividends, and interest payments. Also transaction costs, interest rate margins for lending and borrowing, and charges for short positions may be taken into account.

The manager has preferences given by expected value of an additive utility function $\sum_{t=0}^{T} u_t(c_t)$ determined by consumption c_t over T periods and by the terminal portfolio value c_T. With a constant relative risk aversion $1 - \gamma > 0$, stage t utility function $u_t(c)$ is $\rho_t/\gamma\ c^\gamma$, for $\gamma \neq 0$, and $\rho_t \log c$, for $\gamma = 0$. Utility discounting factors are given by $\rho_t = exp(-\rho t\Delta)$, where ρ is a constant. For node k at time t, denote the utility of consumption c_k by $u_k(c_k)$.

The consumption-investment problem is to find an investment strategy, levels of consumption and terminal wealth, to

$$\max \sum_k \pi_k u_k(c_k) \tag{1}$$

subject to constraints specifying cash balance equations, position dynamics equations and portfolio weight restrictions. The consumption-investment problem is fully stated in (12) of the Appendix, where position dynamics are given by (7),cash balance by (10), and weight restrictions by (11). In order to avoid excessive notation, repetition of mathematical formulation is omitted for the specialized model (1).

To deal with the optimal exercising of options, we append incentive call option as the fourth asset in the problem (1). Then, the single model (1) can be used throughout for numerical analysis. The initial position of options reveals the number of options held for exercising. Additional options in possession are sold initially at market price and the resulting revenue is incremented in the initial exogenous cash flow. For instance, in sections 3 and 5 below, all options are sold initially, while in Section 4 we parameterize the fraction of options sold. For the options held initially, in order to determine optimal exercising together with investment and consumption, we prohibit both short position in the option and an increase in long position; for implementation, see the discussion in the Appendix. The quantity sold is now interpreted as the number of options exercised, and the sales price is interpreted as the payoff of exercising one option.

Assuming that the problem (1) is feasible and no arbitrage opportunities exist, then an optimal solution exists and the optimal consumption stream (c_k) is unique. Furthermore, optimal dual multipliers $\lambda_k = \pi_k u'_k(c_k)$ for cash balance equations are strictly positive and unique.

2.2 Marginal ask price valuation

Consider valuation of an incentive call option on the stock with a maturity of $M \leq N$ years and exercise price X. The marginal ask price for such an option is the price at which the manager is indifferent between selling and not selling a small quantity of options in possession. In this section we introduce the methodology applied in Sections 3–6 for ask price valuation of incentive options. A general development of such methodology is presented in the Appendix, where we derive the valuation results and point out the relationship with arbitrage pricing theory.

Consider increments $\delta = (\delta_k)$ of the exogenous endowment in the cash balance equations, and let $\hat{U}(\delta)$ denote the resulting optimal expected utility. Then the gradient of $\hat{U}(\delta)$ with respect to δ exists at $\delta = 0$, and optimal multipliers λ_k of the cash balance equations yield marginal increments in the optimal expected utility for an increment in cash balance equation of node k. Hence, if an additional δ_k units of cash is provided at node k to relax the cash balance equation, then the optimal expected utility increases approximately by $\lambda_k \delta_k$.

For marginal ask price valuation, the manager is considering to sell an option in a small quantity ϵ. Given a price V for the option, a sales revenue ϵV is received at stage $t = 0$. Then, the marginal ask price V is the minimum unit price at which the manager is willing to sell a small share ϵ of the option.

We consider various types of European and American options. Let $f = (f_k)$ be an option cash flow stream associated with a particular exercising strategy. Given all possible exercising strategies, there is a set F of attainable cash flow streams f. Consider a small share $\epsilon > 0$ of the option. If what is received is worth as much as what is given up (in terms of utility), the marginal ask price $V(f)$ of f satisfies the indifference equation

$$\lambda_0[\epsilon V(f)] = \sum_k \lambda_k[\epsilon f_k].$$

Hence, using optimal marginal utility u'_k of node k at time t, $\lambda_k = \pi_k u'_k(c_k)$ and

$$V(f) = \sum_k \kappa_k f_k, \tag{2}$$

where the state price κ_k are given by

$$\kappa_k = \lambda_k/\lambda_0 = \pi_k u'_k/u'_0 = \pi_k \rho_t (c_0/c_k)^{1-\gamma}. \tag{3}$$

Consequently, the marginal ask price of the option is

$$V = \max_{f \in F} V(f). \tag{4}$$

Optimality conditions for (1) imply that $\kappa_k > 0$ and $\kappa_0 = 1$ in (3). As shown in the Appendix, state prices κ_k constitute arbitrage free state prices. Hence, the marginal ask price valuation is consistent with arbitrage pricing theory. A unique subjective marginal option value V is obtained even if arbitrage free state prices are not unique.

For numerical evaluations in Section 3, we employ the following observations; see Lemma 3 in the Appendix. If the initial positions are zero and the exogenous endowment e_k is zero except at the root node $e_0 > 0$, then the values $V(f)$ in (2) and V in (4) are independent of the initial endowment $e_0 > 0$. If additionally there are no friction costs, then the values $V(f)$ and V are independent of the utility discounting factors ρ_t, and valuation of an option with a maturity of M years is independent of time horizon N, as far as $N \geq M$.

Arbitrage free bounds for option values are uniform applying to all utility functions considered above, and they are independent of the private endowment process. The smallest upper bound $V^+(f)$ and largest lower bound $V^-(f)$ for the value $V(f)$ is obtained by linear programming (see Appendix), and arbitrage free bounds for the option value are given by

$$\max_f V^-(f) \leq V \leq \max_f V^+(f). \tag{5}$$

For a European option maturing at time t, let $\bar{f}$ be the expected value and σ_f the standard deviation of f_k. Define stochastic discounting factors (SDFs) z_k by $\pi_k z_k = \kappa_k$. At time t, let $\bar{z}$ be the expected value and σ_z be the standard deviation of z_k. If ρ_{zf} denotes the correlation coefficient of the option cash flow and the SDF at time t, then by (2), we have

$$V(f) = E\left[z_k f_k\right] = \bar{z}\bar{f} + \rho_{zf}\sigma_z\sigma_f. \tag{6}$$

Given $z_k = \rho_t(c_0/c_k)^{1-\gamma}$, the SDF is proportional to the inverse of relative increase in consumption c_k/c_0 raised to power $1-\gamma$. A high increase in consumption implies that the SDF and the state price are low. Also, if $c_k > c_0$, then the state price decrease as risk aversion increases. If there is a high increase in consumption with a high stock price, then we may expect ρ_{zf} to be negative.

2.3 Model data and implementation for computations

In Sections 3–5 we discuss a number of cases and compare some results with

Ingersoll (2006). Hence, following Ingersoll we use an initial price of the stock $S_0 = 100$, an exercise price $X = 100$ for the call, and a maturity of $M = 10$ years. The time horizon of the model is the maturity of the option. Hence, we set $N = M$. However, the impact of a longer horizon with $N > M$ is discussed. For the case studies of section 6 concerning Fortum and Nokia, data is provided in Table 6. In these cases, we also account for proportional income and capital gains taxes to make the setup as realistic possible.

Data for price processes is summarized in Table 1. Given the step size of Δ years in the model, the annual logarithmic risk free return 0.05 implies a return $r = 0.05\Delta$ for a single period. Similarly, the annual volatility 0.3 of the stock price implies a single period volatility $\sigma = 0.3\sqrt{\Delta}$. Unlike in continuous time analysis, we also need numerical values for ν_m, ν, β and σ_m. Conforming to Ingersoll [13], we assume the CAPM relation $\nu = r + \beta(\nu_m - r)$, which provides ν given ν_m and β. Given ν_m, β is obtained from $\sigma^2 = \beta^2\sigma_m^2 + v^2$, where σ_m is determined such that the optimal level of investment in the stock is zero in the perfect market case. Numerically such value σ_m can be easily evaluated.

In subjective valuation, for comparison with results of Ingersoll [13], we use $\gamma = -2$, for power utility, and $\gamma = 0$, for log utility. The weight limit on stock in manager's investment portfolio is α, where α is 0, 0.1 or 0.5.

We utilize modern numerical analysis for multi-stage stochastic optimization; see Wets and Ziemba [26]. Computations reported in Sections 3–6 are carried out using AMPL with an interior point solver MOSEK; see Fourer, Gay and Kernighan [7]. AMPL is an algebraic modeling language, which makes it easy to implement optimization models such as (1) and (12). AMPL also reads data files specifying numerical values for model parameters and it calls for an optimization code (in our case MOSEK) to compute optimal primal and dual solutions. Precision of numerical optimization, given a particular optimization model, is specified by tolerance parameters, for which we used smallest possible values (10^{-14}) leading to a highest possible precision. For instance, the optimal objective function value is computed with a relative error less than 10^{-10}.

3 Variations in European and American options

We begin the numerical analysis in Section 3.1 by a comparison of European option values obtained from the continuous time model of Ingersoll [13] and dircrtete time models. For $e_k = 0$, for $k > 0$, and omitting friction costs we show how European call values from our discrete time model converge to those of Ingersoll [13]. In Section 3.2 we demonstrate for European options the impact of friction costs: transaction costs, short position charges, and interest rate margins for borrowing and lending. In Section 3.3, American call values are discussed with and without friction costs, dividends on the stock, and a vesting period. A vesting period of τ years, $0 \le \tau \le M$, may apply to the incentive option.

3.1 Basic European options

For comparison with results of Ingersoll [13] we first consider the case without friction costs. With a 10 year maturity and an exercise price 100 we evaluate the marginal ask price for a European call option. Assume the exogenous endowment $e_k = 0$, for $k > 0$, so that salary is excluded from consideration. It is assumed, that that manager considers selling all options in possession[4]. In this case, the marginal ask price is the minimum price at which the manager is willing to sell all options. There are no dividends in this case. We consider two values for the portfolio weight limit $\alpha = 0.1$ and $\alpha = 0.5$, and two levels 1 ($\gamma = 0$) and 3 ($\gamma = -2$) for relative risk aversion. In this case, by Lemma 3 (i)-(ii), the ask price is independent of the initial endowment $e_0 > 0$ and the utility discounting parameter ρ.

For (2), let $\kappa_k = \lambda_k/\lambda_0 = \pi_k z_k$ and $\overline{z} = \sum_{k \in K_T} \kappa_k = \sum_k \pi_k z_k = E_t\left[z_k\right]$ and define $\psi_k = \kappa_k/\overline{z}$. Then, by Lemma 3 (iii), ψ_k is a multinomial distribution, which is obtained from optimal dual multipliers of equations (10) in a single period portfolio problem, and the option value $V(f)$ in (2) is $\overline{z}\sum_{k \in K_T} \psi_k f_k$. The discounting parameter $\overline{z}$ is the subjective risk free discounting of Ingersoll [13], and ψ_k is a subjective risk neutral probability. Perfect market prices are obtained from (5), where the arbitrage free price interval is a single value. Equivalently, in this case, the market prices are obtained with $\alpha = 0$, because the stock weight restriction is not binding.

The marginal ask prices for the European call option are shown in Table 2 for $\alpha = 0.1$ and $\alpha = 0.5$, and $\gamma = 0$ and $\gamma = -2$. The number of time steps T ranges from 4 to 1000. In all cases, the ask price is smaller than the market price. Hence the manager would be willing to sell all options at the market price.

[4] If some options are exercised after time $t = 0$, then the option cash flow interferes ask price valuation; see Section 4.

We observe, that the call values obtained converge at $T = 1000$ to those obtained with the continuous time model. Even with a small number of time steps T the call values deviate from continuous time values by a few percent, at most by six percent. This is a justification of the stochastic programming approach with small T in subsequent illustrations, which show that there are other factors of practical importance, such as market frictions, options held by the manager and risky salary, which result in price deviations of even tens of percents as compared with continuous time results of cases without friction costs and risky private endowments.

	$\alpha = 0.1$		$\alpha = 0.5$		
T	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$	Market
4	49.97	43.73	42.35	23.15	52.88
6	49.23	43.48	41.86	23.15	52.55
8	49.35	43.35	41.58	23.32	52.36
10	49.33	43.40	41.63	23.64	52.34
100	49.49	43.73	41.77	24.60	52.56
1000	49.48	43.75	41.76	24.68	52.57
∞	49.48	43.75	41.76	24.69	52.57

Table 2: European call values without friction costs dividends and salary. The values are ask prices for selling all options in possession at the initial stage. There are no dividends nor salary. Market refers to perfect market and cases $T = \infty$ to continuous time values by Ingersoll [13].

3.2 European options with friction costs

We now modify the previous case introducing friction costs: proportional transaction cost of 0.1 percent for buying and selling, 1 percent interest rate margin between borrowing and lending, and short position cost of 2 percent per annum. For the European call option with a 10 year maturity and an exercise price 100, marginal values are shown in Table 3 for $\alpha = 0$, $\alpha = 0.1$ and $\alpha = 0.5$, $\gamma = 0$ and $\gamma = -2$, and for the number of time steps T ranging from 4 to 8. The utility discounting parameter is $\rho = 0$, implying $\rho_t = 1$, for all t. There are no dividends, there is no salary and no vesting period. For computation of ask prices we use (2). Based on (5), also arbitrage free market prices are shown for some cases. By Lemma 3 (i)-(ii) in the Appendix, the ask price is independent of the initial endowment $e_0 > 0$.

Unlike in the preceding case, even with $\alpha = 0$ the stock weight restriction is binding, and consequently, the weight limit has a small impact on the call value. Comparing the values with those excluding friction costs, we observe a minor impact in case $\alpha = 0$. However, for $\alpha = 0.5$ and $\gamma = -2$, the value decreases by about 15 percent due to friction costs. Even for $\alpha = 0$, a few percent decrease is observed.

Based on (5), arbitrage free market price intervals $[V^-, V^+]$ are computed omitting the weight restriction α on stock and assuming an equilibrium such that an agent is indifferent in investing and not investing a marginal amount in the stock. The market price interval $V^+ - V^-$ is relatively large, about 20 percent of the upper limit. For $\alpha = 0.5$, the call values are below the market price interval. Hence, as above, the manager is willing to sell all options at any market price in $[V^-, V^+]$. However, for $\alpha = 0$ and $\alpha = 0.1$, the ask price is within the market price interval. Hence, depending on the prevailing market price, the manager may or may not be willing to sell all options.

	$\alpha = 0$		$\alpha = 0.1$		$\alpha = 0.5$			
T	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$	V^-	V^+
4	51.53	51.49	50.17	43.92	42.25	19.76	43.30	53.49
6	51.15	51.12	49.75	43.71	41.64	19.60	43.06	53.21
8	50.80	50.91	49.16	43.49	41.37	19.64	42.90	53.05
∞	52.57		49.48	43.75	41.76	24.69	52.57	

Table 3: European call value with friction costs in comparison with continuous time values without friction costs ($T = \infty$) by Ingersoll [13]. The values are ask prices for selling all options in possession at the initial stage. There are no dividends nor salary. The arbitrage free market price interval is $[V^-, V^+]$, which is a single value in the perfect market case.

In addition to option values, solution to the portfolio problem involves the consumption and investment profiles of the manager. Figure 2 shows how the manager's consumption and investment portfolio evolve over time in the power utility case with risk aversion $\gamma = -2$, and portfolio restriction $\alpha = 0.5$. This model has eight periods and 4^8 nodes in the final period. Option values associated with this case are found in the $T = 8$ row of Table 3.

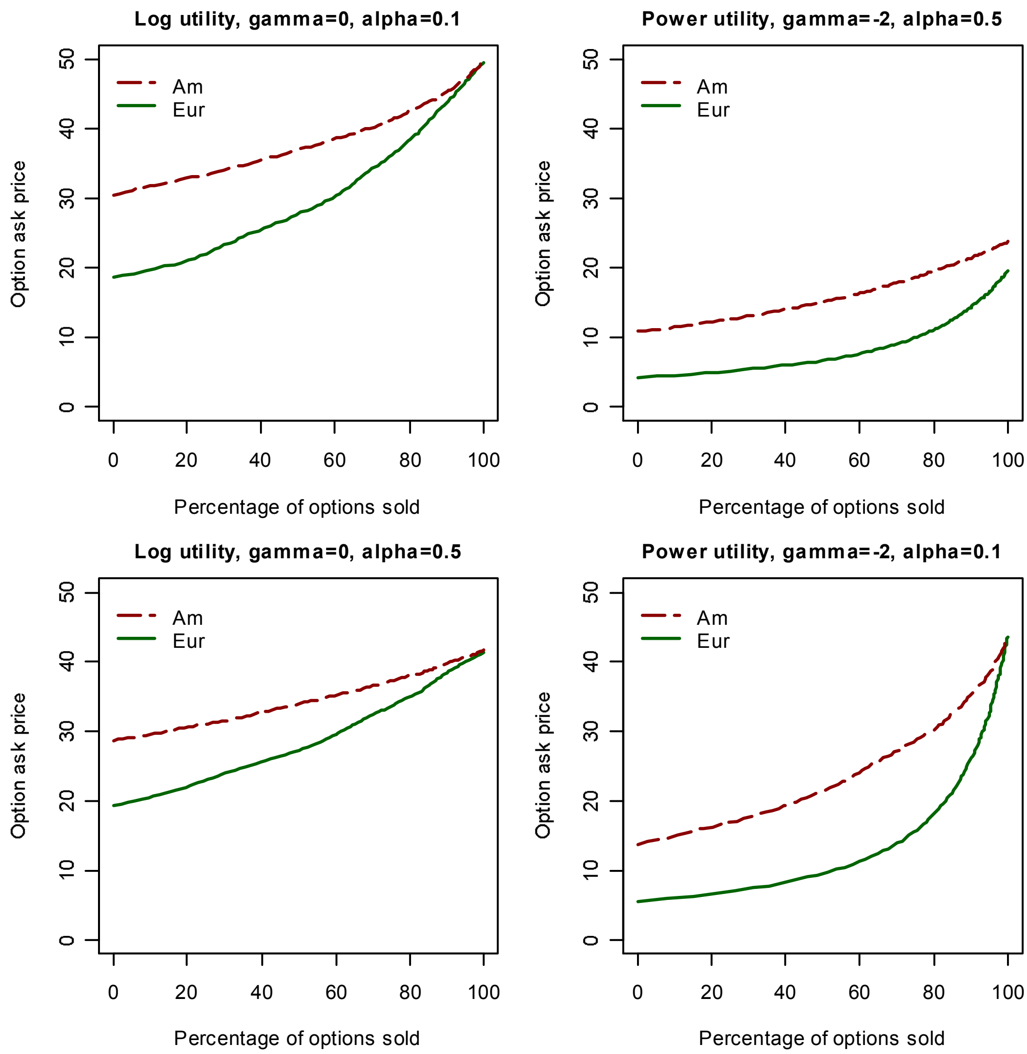

Figure 2: Consumption and investment profiles of a manager with power utility ($\gamma = -2$) and portfolio restriction $\alpha = 0.5$. There is no labor income. In all four panels, solid lines point the mean and dashed lines point mean $\pm$ 1 st.dev.

In Figure 2, the top-left panel shows that under the circumstances the manager is able to increase consumption by 45 % during the 10-year horizon. Variance of consumption increases over time, because actual consumption depends on investment returns. Given the risk-averse nature of this case, the manager is conservative and invests (i.e. saves) about 40 units in the risk-free security. The rest is allocated to market portfolio and to the stock. Portfolio weights converge to zero, because all wealth is consumed at the end, there is no bequest. Continuing with Fig. 2, the bottom-right panel shows that investment to company stock is about as large as risk-free asset, implying that the porfolio constraint is binding and diversification is minimal at the outset. If we compare the profiles shown in Figure 2 to other cases of Table 2, changes have the following character. When risk aversion decreases, the impact on initial consumption is minor, but end-of-horizon consumption increases significantly. Unsurprisingly, the variance of consumption increases as well. These results can be traced to reduced allocation in the risk-free asset, and increased allocation to risky assets.

A distinctive feature of Table 3 is that when portfolio restrictions apply, increasing risk aversion decreases option value. The effect is magnified in the case $\alpha = 0.5$, where half of investment wealth must be allocated to the stock. The reason for low option value is that the portfolio constraint forces the manager to hold an undiversified portfolio that is far from optimal. The undiversified portfolio leads to low expected consumption, which implies high marginal value of consumption, and low asking price of the option, as shown in Table 3.

As indicated by the remark following Lemma 3 (see Appendix), in the frictionless cases of Table 3 the option values remain unchanged if the planning horizon N is extended beyond maturity M of the option. Hence, in cases of Table 3 with friction costs, one might expect a minor impact by extended horizons. To demonstrate this, we made a quick test. The six cases of Table 3 were first computed with $N = M$ and $T = 3$ steps to obtain six call values. Thereafter, the planning horizon is doubled to $N = 20$ and valuation of the same options with maturity $M = 10$ were carried out using a $T = 6$ step model. As a result we obtain two sets of six call values. Comparing these values case by case indicate that the worst case difference is less than one per mille.

3.3 Early exercise, dividends and vesting

Next, we consider cases of Sections 3.1–3.2 modified to allow early exercise of the call option, dividends on stock and a vesting period of τ years. In case dividends are included, we have $q = 0.01\Delta$ and $q_m = 0$ allowing dividends yield on the stock only. In case a vesting period is imposed, we have $\tau = 4$ years. No salary is considered. For valuations we use (2) and (4) with standard backward recursion. Table 3 shows American call values with and without friction costs, each in three cases: $q = 0$ and $\tau = 0$, $q = 0.01\Delta$ and $\tau = 0$, and $q = 0.01\Delta$ and $\tau = 4$. Again, the values are ask prices for selling all options in possession.

As is well known, if there are no dividends nor frictions, then early exercise does not pay off. However, $\alpha > 0$ represents a friction and in this case one may benefit from early exercise even if there are no friction costs nor dividends. For instance, for $\alpha = 0.5$ and $\gamma = -2$, excluding friction costs and dividends, the American call value 26.63 is well above the European call value 23.32. If friction costs are included, then the gain from early exercise increases in comparison with the case without friction costs.

		$\alpha = 0.1$		$\alpha = 0.5$			
q/Δ	τ	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$	V^-	V^+
European options, without friction costs							
0	0	49.35	43.35	41.58	23.32	52.36	
0.01	0	42.43	37.24	35.65	19.65	45.18	
0.01	4	42.43	37.24	35.65	19.65	45.18	
European options, with friction costs							
0	0	49.16	43.49	41.37	19.64	43.17	53.05
0.01	0	42.46	37.31	35.38	16.46	37.89	45.71
0.01	4	42.46	37.31	35.38	16.46	37.85	45.71
American options, without friction costs							
0	0	49.32	43.52	41.79	26.63	52.36	
0.01	0	42.72	38.16	36.69	23.98	45.18	
0.01	4	42.72	38.13	36.65	23.24	45.18	
American options, with friction costs							
0	0	49.16	43.59	41.52	23.93	43.17	53.05
0.01	0	42.70	38.12	36.41	21.79	37.89	45.71
0.01	4	42.70	38.09	36.38	20.48	37.85	45.71

Table 4: European and American call values with and without friction costs, dividends and a vesting period; q = dividend yield on stock, τ = vesting period (years). The values are ask prices for selling all options in possession at the initial stage. The number of steps is $T = 8$. No salary is considered. The arbitrage free market price interval is $[V^-, V^+]$, which is a single value for a

perfect market.

Also Detemple and Sundaresan [6] point out that early exercise may be optimal when market frictions are accounted for, even if there are no dividends. Their intuition is that early exercise increases the manager's utility because it helps to deal with the short-sales constraint. In fact there are two dimensions to this effect. On one hand, early exercise (which may be partial) reduces the need the hedge the incentive stock option. On the other hand, exercising the option increases the manager's liquid wealth, which helps to reduce the suboptimality of constrained portfolio.

If dividends are included, it is well known that one may benefit from early exercise even if the weight restriction on stock is omitted and there are no friction costs. For $q = 0.01\Delta$ and $\tau = 0$, call values are below the values obtained without dividends, because dividends decrease the stock price. The early exercise gain increases due to dividends, as expected.

If both dividends with $q = 0.01\Delta$ and a vesting period of $\tau = 4$ years is included, the resulting the sacrifice from the vesting period is relatively small in comparison with cases with $q = 0.01\Delta$ and $\tau = 0$. American option with $\alpha = 0.5$ and $\gamma = -2$ is an exception, where the call values are reduced significantly.

In Table 4, for all cases where friction costs are omitted or $\alpha = 0.5$, the call values are below the market price interval so that the manager is willing to sell all options at market price. For other cases, the call value is within the market price interval and the manager's willingness to sell depends on the prevailing market price.

All runs of Table 4 were done with $N = M = 10$; i.e. with a planning horizon matching the maturity. The four cases of American options with friction costs, but without dividends and a vesting period, were tested for the impact in option values when the planning horizon is extended. As in Section 3.2, we first compute the four call values with $N = M = 10$ and $T = 3$. Thereafter, the four valuations are made with $N = 20$, $M = 10$ and $T = 6$. Again comparing the four pairs of option values indicates that the worst case difference is less than one per mille.

4 Option ask price functions

Until now we have assumed that if the managers trades, he will sell all his options. In practical situations the managerial option holdings consist of several grants having different times of vesting. Therefore the manager is likely to unload his incentive option position by selling in parts. The effect of partial sales is to change the indifference value of remaining options, because cash flows from exercising are added in the model. Hence, Lemma 3 (i) in the Appendix no longer holds, and price and quantity become dependent in subjective valuation, pointing the need for a *supply function of incentive options*; i.e., the optimal quantity sold as a function of option price.

In option pricing literature supply functions appear rarely, because the usual arbitrage valuation produces only one price that is independent of quantity. It is implicitly assumed that supply function is infinitely elastic, since above the arbitrage price there is unlimited supply, and below the arbitrage price nothing is offered.

In contrast, valuation by indifference implies that price and quantity are no more independent. If only a small share of options are sold at time zero, there is obviously lower initial consumption compared to the case, where all options are sold at that point. Of course, this holds also when 'time zero' is the time of vesting. In addition to initial consumption, the sold quantity affects consumption in all later time points, altering the consumption distributions. It follows immediately that all state prices change as well. For European options, for instance, the indifference value, given by $V(f) = E[z_k f_k] = \bar{z}\bar{f} + \rho_{zf}\sigma_z\sigma_f$, changes because the distribution of stochastic discount factors is affected. Specifically, the SDF variance σ_z^2 plays a certain role, since if nothing is sold at the start, the variance of terminal consumption increases significantly. An important detail is that the correlation ρ_{zf} is usually negative.

Given that we calculate the ask price as function of the sold quantity, the result represents the supply function for incentive options. Figure 3 shows inverse supply functions (price functions) in several cases, which have different risk aversion and portfolio restrictions. The calculations assume that the manager initially has a wealth of e_0 and a given number of options possession. This initial option position is determined such that its value is $e_0/2$ at perfect market price of European options. Further, he considers selling a fraction δ of the options, and the proceeds are invested optimally at time zero. After that no more option sales are allowed. The unsold fraction $1-\delta$ may be exercised either at the end (for European options) or at an arbitrary time point (for American options). The optimal exercise is determined jointly with optimal consumption and investment.

In line with intuition, the price functions of Figure 3 are upward sloping.

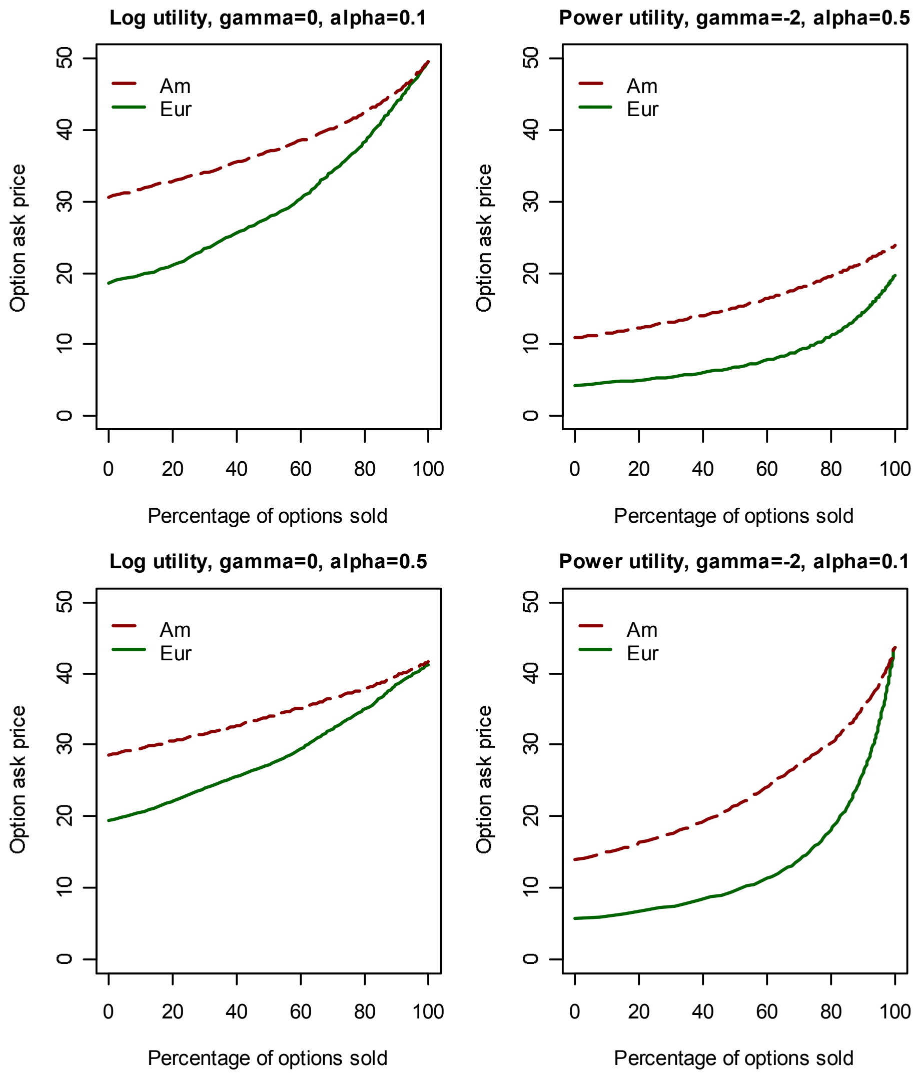

Figure 3: Ask price functions, i.e. inverse supply functions for incentive stock options in four cases, where α is either 0.1 or 0.5 and relative risk aversion $1-\gamma$ is either 1 or 3. Separate price functions are provided for American and European options.

The supply functions are generally more elastic for American options. One explanation for this is that American option holders are able to smooth their consumption by cashing in 'down the road', whereas European option holders have to wait until expiration.

Figure 4 shows explicitly, how consumption smoothing works. In the left panel, we plot the distribution of consumption in the case where only a differential fraction is sold at the start. In this case the options are European, risk aversion is $\gamma = -2$ and the portfolio restriction is $\alpha = 0.5$. The right panel of Fig. 4 plots the case, where all options are sold at the start. It is clear that the consumption path is relative smooth in the latter case, however at the expense of significantly lower expected consumption at the end. Note also the different shapes of consumption distributions; keeping almost all options to expiration results in higher variance of consumption. It leads to higher variance of the SDF, which has a negative effect on the indifference value. The economic interpretation is unequivocal; the consumer dislikes uncertainty.

Finally, Figure 3 may help to clarify the early exercise puzzle, documented by Carpenter [4], among others. Assume a setup where incentive options are not traded, but they may be exercised. A look at the price functions shows that the indifference value decreases if the manager considers exercising only a small fraction, and at some point this indifference value equals the intrinsic value. It follows that it is optimal for the manager to exercise this fraction, given the consumption-investment problem. Again, the intuition for this result can be found in the consumption smoothing effect.

5 Impact of risky labor income

Consider a manager, whose income and investment risks are to some extent tied to the stock price of the employer firm. Then, both income and investment risks simultaneously interact with the portfolio choice of the manager. The situation arises, for instance, if income includes bonuses. First, they contribute to the variance of labor income; see Mercer Group study cited in the introduction. Second, while bonuses are based on some performance measures, it is plausible that financial performance and stock returns are correlated.

Leone, Wu and Zimmermann [20] report that the correlation of CEO cash compensation and stock returns is appr. 0.3 in the S&P Execucomp data for years 1993–2003. In fact, Table 4 of Leone, Wu and Zimmermann [20] shows

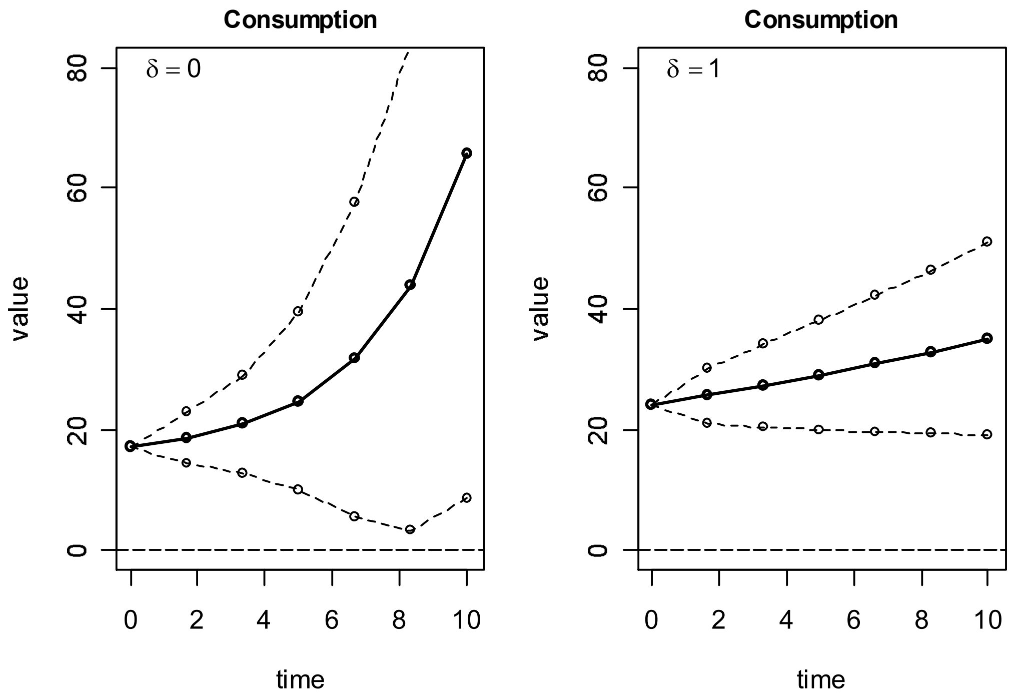

Figure 4: Consumption profiles in two cases, where the manager sells either a small fraction of his options or all of them. In both panels, solid lines point the mean consumption, and dashed lines point mean $\pm$ 1 st.dev.

that cash compensation reacts asymmetrically to stock returns; the impact of stock returns is higher if they are negative. Hence, empirical data supports the notion that managers should consider stock market risks when they value equity-based compensation.

In this section we examine the effect of labor income on valuation of incentive options. We are not aware of any other study with a similar framework. There is a plenty of research on incentive options, and a substantial research stream on lifecycle portfolio choice, but these two do not cross. We calculate option values for American and European calls, with and without friction costs. Motivated by Campbell and Viceira [3, Chapter 6], we rely on three dimensional geometric Brownian motion of labor income, market return and stock return. For the logarithmic increment of the labor income, we assume drift, volatility, and correlations with market and stock return. Denote the drift by ν_l, volatility by σ_l, correlation coefficient with market return by ρ_{ml}, and a correlation coefficient with stock return by ρ_{sl}. For implementation, labor income realizations are adapted to our double binary event tree illustrated in Figure 1. Hence, the logarithmic increments of the labor income are chosen to meet the assumed drift, volatility, and correlation requirements.

In the presence of salary, Lemma 3 (i) no longer holds. Hence, besides the salary, also the initial liquid holdings influence subjective option valuation. In the examples below, we have chosen the initial liquid holdings equal to the initial annual salary.

Subsequently, five cases A–E are demonstrated. The income process is parameterized as follows: (A) labor income is deterministic with a drift $\nu_l = 0.03$; (B) with $\nu_l = 0.03$, the income volatility is increased to $\sigma_l = 0.10$ while $\rho_{ml} = 0$ and $\rho_{sl} = 0$ so that labor income volatility represents an idiosyncratic risk; (C) with $\nu_l = 0.03$ and $\sigma_l = 0.10$, correlation with stock return is increased to $\rho_{sl} = 0.40$ while the correlation with market return remains at $\rho_{ml} = 0$; (D) both correlations are positive with $\rho_{ml} = 0.20$ and $\rho_{sl} = 0.40$; and (E) the drift is reduced to $\nu_l = 0$, leaving other parameters to levels above: $\sigma_l = 0.10$, $\rho_{ml} = 0.20$ and $\rho_{sl} = 0.40$.

					$\alpha = 0.1$		$\alpha = 0.5$	
Case	ν_l	σ_l	ρ_{ml}	ρ_{sl}	$\gamma = 0$	$\gamma = -2$	$\gamma = 0$	$\gamma = -2$
European options								
A	0.03	0.00	0.00	0.00	51.75	51.75	50.28	49.99
B	0.03	0.10	0.00	0.00	51.52	55.14	50.29	52.23
C	0.03	0.10	0.00	0.40	44.02	27.12	43.80	28.70
D	0.03	0.10	0.20	0.40	46.61	32.96	45.99	33.76
E	0.00	0.10	0.20	0.40	47.07	34.36	45.83	32.34
no salary					49.16	43.49	41.37	19.64
American options								
A	0.03	0.00	0.00	0.00	51.75	51.79	50.32	50.07
B	0.03	0.10	0.00	0.00	51.52	55.15	50.36	52.53
C	0.03	0.10	0.00	0.40	44.06	28.82	43.87	30.04
D	0.03	0.10	0.20	0.40	46.62	33.53	46.02	34.17
E	0.00	0.10	0.20	0.40	47.07	34.74	45.87	33.08
no salary					49.16	43.59	41.52	23.93

Table 5: European and American call values with salary and friction costs. For the salary process, ν_l is drift, σ_l is volatility, ρ_{ml} is correlation coefficient with index and ρ_{sl} is correlation coefficient with stock price. The number of time steps is $T = 8$. There are no dividends. The values are ask prices for selling all options in possession at the initial stage. For comparison, call values without salary are shown.

Valuation results for the five cases A − E are presented in Table 5. Values of European and American calls are shown for each case with friction costs, and with variations in the stock weight restriction α and in the risk aversion parameter γ.

In case A with a deterministic salary, the option values are relatively insensitive to changes in α and γ, and the values are above those in Table 4 obtained without salary. This is explained by a decreased absolute risk aversion due to increased consumption. Comparing cases A and B shows how idiosyncratic labor income risk affects option values. With moderate risk aversion ($\gamma = 0$), idiosyncratic risk has almost no effect on option value. But if we increase risk aversion to $\gamma = -2$, idiosyncratic risk actually increases the option value.

Correlated labor income risk is treated in cases C and D of Table 5. Compared to cases A and B, in cases C and D we add correlations with stock and market returns, respectively. By and large, correlating labor income with stock and market returns reduces incentive option values. If we assume power utility, setting cases C and D against the case without salary reveals a 24–38 percent reduction in incentive option value, if we impose minimal stock weight of 0.1. If we do this comparison using log utility, the effect of correlated risks on labor income is significantly smaller.

A less obvious effect is that adding a correlated salary has a positive effect on incentive option values, if the stock weight restriction is tightened to 0.5. The underlying intuition is clear; adding the salary in fact helps the manager to diversify his portfolio, since the correlations are far from perfect ($\rho_{ml} = 0.2, \rho_{sl} = 0.4$). Incentive options are more valuable as part of a better diversified portfolio. If labor income risk is idiosyncratic or absent, the positive effect on option value is even more pronounced. Somewhat surprisingly, in case E the reduction of salary drift ν_l to zero has a relatively minor impact in comparison with case D.

To illustrate the consumption and investment profiles obtained from optimization, Figure 5 shows case D with $\alpha = 0.5$ and $\gamma = -2$. For comparison, Figure 2 shows the profile with $\alpha = 0.5$ and $\gamma = -2$ without salary. As shown in Table 4 the option values with salary are well above those without salary. For European options the increase in value is 72 %. In Figures 2 and 5, growth rates of consumption are similar; however there is a major difference in levels. We also remark that saving at the start turns to borrowing, when labor income is included, which can be checked by comparing the risk-free asset allocations of Figures 2 and 5. Of course, we are not arguing that introducing labor income would unconditionally induce levered portfolios. Kahl, Liu and Longstaff [14] solve the portfolio problem of a manager who has restricted stock (but no options or labor income) and also find that in some cases it is optimal to take levered positions in the market portfolio. Their explanation, which we share, is that stock market risk is used to hedge various undiversifiable risks.

We will now employ the stochastic discount factor framework to relate two seemingly distant matters: consumption *cum* labor income and subjective option pricing. In order to explain the large price differences in Table 5, one has to understand that labor income changes the *distribution* of state prices. Some analytics are provided by the valuation equation (6). For terminal nodes k, the SDF is $z_k = \rho_t(c_0/c_k)^{1-\gamma}$ so that the state price in (16) is $\kappa_k = \pi_k z_k$, where π_k is the probability of node k. If f_k is the cash flow from the European option, repetition of (6) yields for the option value $V(f) = \bar{z}\bar{f}+\rho_{zf}\sigma_z\sigma_f$, where $\bar{z}$ and $\bar{f}$ are expected values of z_k and f_k, σ_z^2 and σ_f^2 are variances, and ρ_{zf} is the correlation coefficient. To see how the option value changes if salary is removed from case D, we observe that the term $\bar{z}\bar{f}$ increases as $\bar{f}$ remains constant at 156.23 and the subjective discount factor $\bar{z}$ increases from 0.66 to 0.89. In the covariance term the correlation ρ_{zf} remains approximately unchanged at –0.32. Because the volatility of option cash flows, $\sigma_f = 272.33$, is the same in both cases, an increase in the volatility of the SDF σ_z from 0.79 to 1.39 remains the explanation of the price difference. To interpret this, note that consumption with salary in Case D is much higher than in case without salary. High consumption implies low marginal utility, a small value for the SDF z_k, and consequently, a small variance σ_z^2. The correlation ρ_{zf} is negative, because for a high stock price at node k we expect a large cash flow f_k and a high consumption c_k, which leads to a small marginal utility and hence a small SDF z_k.

Given the roles of the SDF variance and correlation with option payoffs, we emphasize that making inferences on valuation requires knowledge of the joint distribution of SDF and option payoffs. Knowing the changes in means ($\bar{z}$ and $\bar{f}$) is not sufficient to conclude how $V(f)$ changes.

The runs of Table 5 are based on $N = M = 10$ years so that the planning horizon matches option maturity. Similarly as in Section 3, we study the impact in call values due to doubling the planning horizon. In all 40 cases of European and American options of Table 5, we compute the call values, first, with $N = M = 10$ and $T = 3$, and thereafter, with $N = 20$, $M = 10$ and $T = 6$. Comparing the pairs of option values indicates the following. For the 20 cases with $\gamma = 0$, the worst case difference less than two per mille. For the 20 cases with $\gamma = -2$, the largest difference is 1.6 percent.

6 Case studies

In this section, we show how practical managers could benefit from our model. This is done by fitting the model to incentive stock option programs in two case companies: Fortum, a major Nordic power company, and Nokia, a leading producer of mobile phones and networks. These case studies are made possible by the fact that incentive options in Fortum and Nokia are actually traded in the Helsinki Stock Exchange. For both companies there also exist active stock and ordinary option markets. In brief, the practice is that after a vesting period, incentive options are quoted in the exchange, and anyone can trade them. In our cases, the options are American in the sense, that after the vesting period they can be freely exercised subject to insider trading restrictions. More detailed description of the incentive options market and trading can be found in Ikäheimo, Kuosa and Puttonen [12].

Our approach to demonstrate our option valuation is to calibrate the market model with empirical data. The parameters are listed in Table 6. The option programs 2001AB of Fortum and 2002AB of Nokia are considered. The former matures in May 2007 and the latter at in December 2007. Exercise prices of these programs were initially endogenous, but at the end of the vesting periods they are fixed to values shown in Table 6.

Due to privacy considerations, personal data, such as salary, wealth, etc. do not relate to any of the managers which we interviewed. For demonstration, we assume that the manager receives labor income and has some initial wealth. Further, we assume that labor income is risky. Risk preferences of the manager are given by log utility. The manager is allowed to trade in market portfolio and stock with the restriction that short positions in the stock are not allowed.

Compared to earlier examples, we add two modifications in the examples of Table 6. First, we set the time preference coefficient for terminal wealth to $\rho_T = 15$ to ensure that adequate wealth is retained at the end of horizon. Second, we add income and capital gains taxes to make the setup as realistic possible.

In case of Fortum, we look at subjective valuation dated back to October 2005, when the options were deep in the money. In case of Nokia, we think

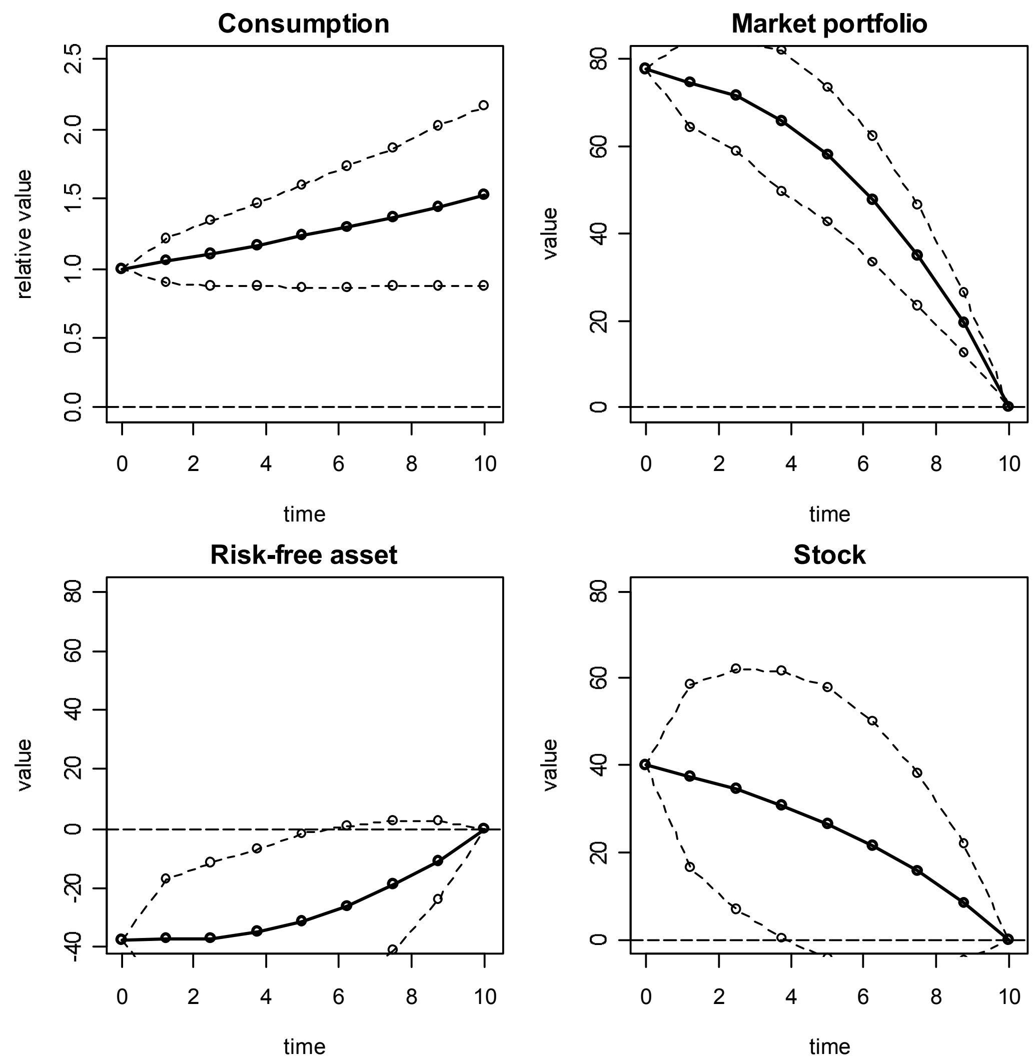

Figure 5: Consumption and investment profiles of a manager with power utility and portfolio restriction. The manager receives labor income that is correlated with market and company stock returns. The correlations are 0.2 and 0.4, respectively. In all four panels, solid lines point the mean and dashed lines point mean ± 1 st.dev.

of valuation in May 2006, when the options were at the money. Hence, the maturities are 1.5 and 1.6 years, for Fortum and Nokia, respectively. The length of time horizon in the discrete time model is set to to maturity and divided in six three months periods. The intuition is that given the market regulation, managers usually find it prudent to trade in the stock and options only when interim reports are disclosed, i.e. once in a quarter.

We consider the manager selling a fraction δ of the options in possession. In table 6, δ is the optimal share given market price of the option. For Nokia, $\delta = 0.4$ is optimal and at this point the subjective option value 2.3 euro is the same as the market price shown for May 2006. In case of Fortum, $\delta = 0$ indicating that the manager would not be willing to sell. In fact, actual trade data of October 2005 indicates a low trading volume for Fortum incentive options.

In order to gain practical insights we interviewed executives who have received the options of Table 6. We were especially interested in how managers evaluate incentive options. The interview revealed that a manager tends to estimate option value crudely as the intrinsic value; i.e., the difference between current stock price and exercise price. Naturally, the time value appears difficult to assess. Hence, we conclude that decision aid based on models such as ours, can be valuable if the time value relative to the intrinsic value is significant; see the Nokia case, for instance.

Because managers tend to sell their options when they are deep in the money, parallel quotes are generally hard to find. Usually the bid-ask spread is wide, and when the manager wants to sell, he turns to market makers for a quote. However, given the good liquidity of both Fortum and Nokia stock, market makers readily provide quotes for any size.

		Fortum	Nokia
ESO	option program (maturity)	2001AB (5/07)	2002AB (12/07)
	exercise price	3.40 e	17.9 e
	time to maturity (years)	1.5	1.6
	market price of ESO (date)	11.7 e (10/05)	2.3 e (5/06)
Assets	risk free rate	3.5 %	3.5 %
	index: drift	6 %	6 %
	volatility	20 %	20 %
	stock: price (date)	14.9 e (10/05)	17.9 e (5/06)
	drift	6 %	6 %
	volatility	20 %	22 %
	dividend	1.0 %	1.0 %
	beta	0.9	1.0
Private	initial wealth	1.0 Me	1.0 Me
	salary: initial (annual)	0.2 Me	0.2 Me
	drift	3 %	3 %
	volatility	10 %	10 %
	initial number of ESOs (1000)	50	300
	relative risk aversion	1	1
	time preference coefficient	15	15
Model	time periods	6	6
	transaction cost	0.1 %	0.1 %
	interest rate spread	1 %	1 %
	short position cost	5 %	5 %
	no short position on stock		
	income tax	50 %	50 %
	capital gain tax	30 %	30 %
Results	fraction sold initially	0 %	40 %
	option value	11.7 e	2.3 e
	consumption (annual)	0.29 Me	0.28 Me
	terminal wealth	1.12 Me	1.21 Me

Table 6: Data for the case studies.

Given the market regulation, managers of both companies found trading in the stock and options feasible only after quarterly reports are disclosed. The representatives stressed that management sketches the firm's future by outlining scenarios and such private views can have a major impact in subjective assessment of the option value. The scenarios may built on different realizations of various risks and they are discrete by nature. For instance, for an electricity firm different scenarios could refer to changes in regulation or decisions taken by competition authorities, since the firm was actively making acquisitions. When the management think about incentive options, the intuition is that if positive scenarios are realized, equity values increase, and the options will be valuable. Incentive schemes enforce the managerial effort to reach positive outcomes for the firm. Hence, this is in line with the setup of classic principal-agent models, where incentives are used to align the interests of managers and owners.

Based on the above, in terms of modeling, managers tend to apply subjective parameters for valuation. Risk aversion is often mentioned in this context, but also the stock price drift and volatility are relevant. Specifically, stock price drift is a relevant parameter in subjective pricing. To illustrate this with sensitivity analysis, consider stock drift increments of Nokia by one percent up and down; i.e., changes from 6 % to 7 % and 5 %. Then the relative changes in subjective option values are 3.5 % and -5.7 % for drift increase and decrease, respectively.

7 Conclusions

A manager's subjective value of an incentive option is the ask price at which the manager is indifferent between selling and not selling an option. While such valuation appears consistent with arbitrage pricing theory, it has the merit that the valuation principle is easy to explain to managers and a unique option value is obtained even in case of an incomplete and imperfect market. The underlying discrete time and discrete state stochastic processes are not restricted. Standard optimization methods are readily available for valuation, and consequently, the level of sophistication in option valuation is modest.

Our analysis indicates that friction costs and labor income can have a major impact in subjective option values. The effect of labor income on option values can be positive or negative, when labor income and stock market risks are correlated. Furthermore, we develop supply functions for the incentive options. If the manager considers selling only a fraction of options in possession and exercising the rest, then the ask price for a fraction can be significantly lower than the ask price for selling all options. Finally, we study ESOs in two case companies calibrating the model to actual market data. Interviews with managers revealed, for instance, that discrete-time analysis is supported due to insider trading restrictions.

A Appendix: Marginal indifference valuation

We adopt the bid/ask price valuation from Kallio [15] with some modifications and new results. We begin by formulating a suitable multi-stage portfolio model, and discuss the consequences of no arbitrage. Thereafter we state properties of our portfolio model, present the valuation results, and point out relationships with arbitrage pricing theory.

In the discrete time approach, the time span of N years is subdivided into T periods defined by stages $t = 0, 1, \ldots, T$. The periods are of equal length. An index $t > 0$ also refers to a period between stages $t-1$ and t.

An event tree specifies the probability measure and filtration describing how information is revealed. The price processes of securities, dividend processes, as well as private exogenous endowment processes of the manager, e.g. salary, are adapted to the event tree. Let $k \geq 0$ denote a node of the event tree with $k = 0$ referring to the root. Let k_- denote the predecessor of node k, for $k > 0$, and let K_t be the set of nodes at time t. Hence K_T is the set of terminal nodes. Node k appears at stage $t_k \in \{0, 1, 2, \ldots, T\}$. For the root, $t_0 = 0$, and for nodes $k \in K_t$, $t_k = t$. For $k > 0$, we assume $t_{k_-} = t_k - 1$ for the predecessor node k_-. Let J_k denote the set of successor nodes of k. Hence, for all $j \in J_k$, we have $j_- = k$, and for terminal nodes $k \in K_T$, J_k is empty. The probability of attaining node k is $\pi_k > 0$, for all k, and $p_j = \pi_j/\pi_k$ is the conditional probability of node $j \in J_k$ given k.

An option specifies a set of possible actions, and a choice of action yields a cash flow f_k, for all nodes k. We denote the stochastic cash flow stream by a vector $f = (f_k)$ and define an option by a set F of feasible choices f. We assume that the choice set F is nonempty, closed and bounded.

Such options may refer to various types of European, American or Asian options, for instance. The choice of an action above refers to a choice of a feasible exercising strategy. Such strategy defines a unique exercising action for all contingencies specified by the event tree. Hence, an action determines uniquely a stochastic payoff stream (f_k). For example, for a European call option with a strike price X, a feasible (not necessarily optimal) strategy is to wait until maturity and then exercise irrespective of the underlying asset price. Another strategy would be the usual one: exercise at maturity if the underlying asset price exceeds X.

Consider finitely many assets i. These assets may refer to interest rate instruments, stock of companies, commodities market, financial derivatives, real estate, etc. Let $\tilde{P}_t$ denote the stochastic vector of prices $\tilde{P}_{ti}$ of all such assets i at stage t. A risk free asset $i = 0$ is included among assets. For this asset, $\tilde{P}_{t0} = 1$, and for simplicity, we assume the total return R over a single period is constant. The realizations of the stochastic price vector $\tilde{P}_t$ are defined in our event tree. For each node k, P_k is the vector of prices at node k.

The vector $y_k^+ \geq 0$ denotes the asset quantities bought and the vector $y_k^- \geq 0$ the quantities sold at node k, for all k. The vector y_k^+ is interpreted as an increase in long positions or as a reduction in short positions and y_k^- is a reduction in long positions or as an increase in short positions.

Let $x_k = x_k^+ - x_k^-$ denote the vector of positions; i.e., quantities held in each instrument at node k, with $x_k^+, x_k^- \geq 0$ referring to long and short positions, respectively. Initial positions $x_{0-} = x_{0-}^+ - x_{0-}^- = \bar{x}$ are fixed. At terminal nodes all positions are closed so that $x_k^+ = x_k^- = 0$, for $k \in K_T$.

The quantities held at node k with initial conditions, for $k = 0$, and closing conditions, for terminal nodes k, satisfy

$$x_k^+ - x_k^- - x_{k_-}^+ + x_{k_-}^- - y_k^+ + y_k^- = 0, \tag{7}$$

$$x_{0_-}^+ - x_{0_-}^- = \bar{x} \tag{8}$$

and

$$x_k^+ = x_k^- = 0 \quad \forall\, k \in K_T. \tag{9}$$

Price vectors P_k^+ of buying and P_k^- of selling include non-negative proportional transaction costs, such that $P_k^- \leq P_k \leq P_k^+$. Transaction costs of short selling are assumed the same as the transaction costs of reducing long positions, and transaction costs of reducing short positions are assumed the same as transaction costs of buying. If there are no transaction costs, then $P_k^- = P_k = P_k^+$. If an asset i cannot be bought at node k, we define $P_{ki}^+ = \infty$, and if it cannot be sold, we have $P_{ki}^- = -\infty$.

We also include dividends, charges for short positions, and periodic interest payments for lending and borrowing with an interest rate spread between borrowing and lending. For each node k, define vectors D_k, D_k^+ and D_k^- of proportional dividend and interest payments as follows. Let $D_k \geq 0$ denote the vector of frictionless proportional yield; i.e., interests at market rates or nominal dividends. Then, the frictionless yield $D_k x_{k_-}$ at node k is determined by the position x_{k_-} taken at the preceding node k_-. For long positions, D_k^+ is the frictionless vector D_k subtracted by friction costs, such as interest rate margin of lending. For short positions, D_k^- is the vector D_k added by friction costs, such as shorting costs and interest margins of borrowing. If long position is prohibited for asset i, we define $D_{ki}^+ = -\infty$. Similarly, short positions are excluded with $D_{ki}^- = \infty$. For all nodes k, we assume nonnegative friction costs so that $D_k^+ \leq D_k \leq D_k^-$. For the risk free asset $i = 0$, the lending rate is D_{k0}^+ and the borrowing rate D_{k0}^- with $D_{k0}^+ \leq R \leq D_{k0}^-$. If there are no friction costs, then $D_k^+ = D_k = D_k^-$.

In Sections 3–5, an asset i refers to an incentive call option with an initial position $\bar{x}_i \geq 0$. In order to determine optimal exercising together with investment and consumption, we define $D_{ki}^- = \infty$ and $P_{ki}^+ = \infty$ to prohibit short position in the option and an increase in long position, respectively. The payoff of exercising one option at node k is P_{ki}^-, and y_{ki}^- is the number of options exercised. There is no additional return so that $D_{ki}^+ = 0$.

For each node k, let c_k denote consumption and let e_k denote a private exogenous endowment of the manager. Given that taxes are not considered in our model, the net cash balance equations, for all k, is

$$c_k + P_k^+ y_k^+ - P_k^- y_k^- - D_k^+ x_{k_-}^+ + D_k^- x_{k_-}^- = e_k. \tag{10}$$

We also consider subjective portfolio restrictions, for $k \notin K_T$, given by

$$E_k(x_k^+ - x_k^-) \leq 0, \tag{11}$$

where E_k is a matrix. Such restrictions may set bounds on portfolio weights and prohibitions of short or long positions, for instance. Alternative formulations in place of (11) can be introduced in a straightforward manner to restrict additionally, for example, relative changes in each position over a single period.

The manager has preferences given by expected value of an additive utility function of consumption stream over $T + 1$ of stages. Let $c = (c_t)$ denote consumption of stage t, $t = 0, 1, \ldots, T$. Then, the utility function is $\sum_{t=0}^{T} u_t(c_t)$, where $u_t(c_t)$ is the stage t utility function. For node k, we denote $u_{t_k} = u_k$, so that the utility of consumption c_k is $u_k(c_k)$.

The consumption-investment problem problem is to find an investment strategy $x_k^+, x_k^- \geq 0$, $y_k^+, y_k^- \geq 0$, and levels of consumption c_k, for all k, to

$$\max \sum_k \pi_k u_k(c_k) \quad s.t. \quad (7) - (11). \tag{12}$$

We assume no arbitrage opportunities exist in the event tree. Formally, we assume that there is no homogenous solution x_k^+, $x_k^- \geq 0$, y_k^+, $y_k^- \geq 0$ and c_k of (7) - (10), satisfying (11), such that $c_k \geq 0$, for all k, and $c_k \neq 0$, for some k.

As shown by Kallio and Ziemba [17], there are no arbitrage opportunities if and only if there exists prices $\kappa_k > 0$, μ_k and $\nu_k \geq 0$ for equations (7), (10) and (11), respectively, satisfying

$$-\nu_k E_k + \sum_{j \in J_k} (\kappa_j D_j^+ + \mu_j) \leq \mu_k \leq -\nu_k E_k + \sum_{j \in J_k} (\kappa_j D_j^- + \mu_j) \quad \forall\, k \not\in K_T, \quad (13)$$

and

$$\kappa_k P_k^- \leq \mu_k \leq \kappa_k P_k^+ \quad \forall\, k. \quad (14)$$

Note that we always can scale prices such that $\kappa_0 = 1$ to obtain state prices. In the perfect market case with position constraints (11) omitted, $\mu_k = \kappa_k P_k$, by (14), and (13) yields the familiar result $\kappa_k P_k = \sum_{j \in J_k} \kappa_j (P_j + D_j)$.

Let C denote the set of feasible (attainable) consumption streams $c = (c_k)$ for (12) and let $U(c)$ be the expected utility given a consumption stream $c \in C$. Then (12) is restated as $\max_{c \in C}\ U(c)$. The valuation results below build on the following lemma.

Lemma 1 *Assume that an optimal solution exists for the problem (12) with a consumption stream $c = (c_k)$. Assume that stage t utility function u_t is increasing, strictly concave and differentiable. Then the optimal consumption stream c is unique. Furthermore, the optimal multiplier vector $\lambda = (\lambda_k)$ for (10) is strictly positive and unique, and optimal multipliers μ_k for (7) and $\nu_k \geq 0$ for (11) satisfy (13) - (14) with $\kappa_k = \lambda_k$.*

Proof: Based on standard optimization theory (see e.g., Mangasarian [21]), the optimal consumption stream c is unique, because C is a convex set and $U(c)$ is strictly concave in C. Furthermore, optimality conditions imply existence of dual multiplier vectors $\lambda = (\lambda_k)$ for (10), μ_k for (7) and $\nu_k \geq 0$ for (11) satisfying (13) - (14) . Finally, $\lambda_k = \pi u'_{t_k}(c_k)$, which is strictly positive by assumption and unique because c_k is unique. $\square$

Optimal multipliers λ_k in Lemma 1 yield marginal increments in the optimal expected utility given an increment in cash balance equation of node k; i.e., if an additional δ_k units of cash is provided at node k to relax the cash equation (10), then the optimal expected utility increases approximately by $\lambda_k \delta_k$.

Let C denote the set of feasible (attainable) consumption streams $c = (c_k)$ for (12) and let $U(c)$ be the expected utility given a consumption stream $c \in C$. Then (12) is restated as $\max_{c \in C} \; U(c)$. Because C is a convex set and $U(c)$ strictly concave in c, the optimal consumption stream c is unique. Consequently, the optimal multipliers $\lambda_k = \pi_k u'(c_k)$ are unique. The following result proves useful for marginal bid/ask price valuation[5].

Lemma 2 *Assume that $U(c)$ is concave and differentiable and an optimal solution exists for (12) with an optimal consumption stream c^* in the interior of the domain of U. Increment the exogenous endowment vector in (12) by $\delta = (\delta_k)$. Assume that an optimal solution exists for $\max_{c \in C} \; U(c + \delta)$, for all δ in some open neighborhood of $\delta = 0$, and let $\hat{U}(\delta)$ denote the optimal expected utility. Then the gradient of $\hat{U}(\delta)$ with respect to δ exists at $\delta = 0$.*

Proof. By assumption, there is $\epsilon > 0$ such that $c^* + \delta$ is a feasible consumption stream, for all δ such that $\| \delta \| < \epsilon$. Hence, we obtain a lower limit $U(c^* + \delta) \leq \hat{U}(\delta)$. For $\delta = 0$, let $\lambda = \nabla U(c^*)$ denote the optimal multiplier vector. Then an upper limit is given by $\hat{U}(\delta) \leq \hat{U}(0) + \lambda\delta$. The assertion follows, because both limits are differentiable with respect to δ and they coincide at $\delta = 0$, and $\nabla_\delta \hat{U}(0) = \lambda$ with $\lambda = \nabla_c U(c^*)$. ■

For marginal bid/ask valuation, one is considering to buy or sell an option in a small quantity ϵ. Given a price V for the option, a buying cost or sales revenue ϵV is applied at stage $t = 0$. Then, the bid price V is the maximum price the manager is willing to pay for the option. Similarly, the ask price is the minimum price at which the manager is willing to sell the option.

Let $f = (f_k) \in F$ be an option cash flow stream associated with a particular exercising strategy. Consider a small share $\epsilon > 0$ of the option. Let ϵ approach zero and define the marginal ask price $V(f)$ as the limiting price at which the manager is indifferent between selling and not selling the share ϵ. Employing Lemma 2, such a limit $V(f)$ is obtained from the indifference equation $0 = \lambda_0[\epsilon V(f)] - \sum_k \lambda_k[\epsilon f_k]$ with optimal dual multipliers λ_k for (12). Hence, the marginal ask price of f is

$$V(f) = \sum_k \kappa_k f_k, \tag{15}$$

[5] The authors are indebted to Teemu Pennanen for suggesting a simple argument for the proof.

where the state prices are given by

$$\kappa_k = \lambda_k/\lambda_0, \tag{16}$$

and the marginal ask price of the option is

$$V = \max_{f \in F} V(f). \tag{17}$$

Because F is nonempty and compact, and $V(f)$ is linear in f, the maximum in (17) is attained. Note that due to valuation at the margin, the bid price equals the ask price. Optimality conditions for (12) imply that $\kappa_k > 0$ and $\kappa_0 = 1$ in (16) satisfy (13) - (14) with some μ_k and $\nu_k \geq 0$. Hence, the marginal bid/ask price valuation is consistent with arbitrage pricing theory. A unique subjective marginal option value V is obtained even if arbitrage free state prices are not unique. We proceed by representing the valuation equation (15) in an asset pricing framework.

Denote $z_k = \kappa_k/\pi_k$. Then, using standard finance terminology, at each stage t, the vector (z_k) associated with nodes $k \in K_t$ at stage t forms stochastic discount factors (SDF) for time t. The standard properties of SDF hold: [i] the expected value of SDF is the reciprocal of the subjective risk-free rate, and [ii] the expected value of discounted returns is equal to one (cf. Campbell and Viceira [3], pp. 38–39). The first property holds because the value of a riskless claim that pays one euro in all states at time t must be equal to one divided by the subjective risk-free return R_t. With $f_k = 1$, for $k \in K_t$, and $f_k = 0$, otherwise, (15) yields $V(f) = \sum_{k \in K_t} \kappa_k \cdot 1 = E_t[z_k] = 1/R_t$, where E_t refers to expectation with respect to node probablities π_k, $k \in K_t$. Hence, the subjective risk free discounting factor $1/R_t$ is $\sum_{k \in K_t} \kappa_k$. To see that the second property holds, let $R_{0,k} = f_k/V(f)$ denote the return on claim f in state k, and rewrite (15) as

$$\sum_k \kappa_k f_k/V(f) = \sum_t \sum_{k \in K_t} \kappa_k R_{0,k} = \sum_t E_t[z_k R_{0,k}] = 1. \tag{18}$$

The following proposition proves useful for some numerical evaluations; see Section 3.

Lemma 3 *Assume that an optimal solution for (12) exists. For exogenous endowments e_k, assume $e_0 > 0$ at the root node and $e_k = 0$ otherwise. For the initial positions, assume $\bar{x} = 0$. For all t, let stage t utility function $u_t(c_t)$ be $\rho_t/\gamma\ c_t^\gamma$, for $0 \neq \gamma < 1$, and, $\rho_t \log c_t$, for $\gamma = 0$, with a utility discounting factor*

$\rho_t > 0$, for all t. Then

(i) the state prices $\kappa_k = \lambda_k/\lambda_0$ in (16) and the value $V(f)$ in (15) are independent of the initial endowment $e_0 > 0$.

If additionally there are no friction costs and single period logarithmic price increments as well as proportional dividend and interest yields are independent of time and state, then

(ii) the state prices κ_k and the value $V(f)$ in (15) are independent of the utility discounting factors ρ_t,

(iii) for optimal dual multipliers λ_k of equations (10), the set $\{\lambda_j/\lambda_k \mid j \in J_k\}$ is the same for all $k \notin K_T$, and

(iv) the optimal investment strategy is fix-mix; i.e., the vector of optimal portfolio weights is the same for all $k \notin K_T$

Proof. For $e_0 = 1$, let c_k, $x_k = (x_k^+, x_k^-)$ and $y_k = (y_k^+, y_k^-)$, for all k, denote an optimal solution for (12), let $\lambda_k > 0$ denote the optimal dual multipliers for equations (10), and let U_0 be the optimal expected utility. Then for any $e_0 > 0$, with dual multipliers upgraded to $e_0^{\gamma-1}\lambda_k$ it follows that the optimal solution is $e_0 c_k$, $e_0 x_k$ and $e_0 y_k$, and the state prices κ_k in (16) and value $V(f)$ in (15) are independent of e_0, concluding (i). Furthermore, to aid the proof of (iii) below, the utility $u_k(e_0 c_k)$ of node k is $e_0^\gamma u_k(c_k)$, for $\gamma \neq 1$, and $\rho_t \log e_0 + u_k(c_k)$, for $\gamma = 0$. Consequently, optimal expected utility is $e_0^\gamma U_0$, for $\gamma \neq 0$, and $\sum_t \rho_t \log e_0 + U_0$, for $\gamma = 0$.

To show (ii), we restate problem (12) as follows. Let $x_k = x_k^+ - x_k^-$ and $y_k = y_k^+ - y_k^-$. For all k, (7) yields $y_k = x_k - x_{k-}$, (10) states $c_k + P_k x_k - P_k x_{k-} = e_k$, and problem (12) becomes

$$\max\{\sum_k \pi_k u_k(c_k) \mid c_k + P_k x_k - P_k x_{k-} = e_k, \qquad (19)$$
$$E_k x_k \leq 0,\ x_{0-} = 0,\ x_k = 0\ \forall k \in K_T\ \}$$

If $\rho_t = 1$, for all t, let x_k and c_k, for all k, be the optimal for (19) with dual multipliers $\lambda_k > 0$ for the cash balance equations. For all t, it follows from (i) that $P_k x_k = w_t c_k$, for some constant w_t independent of $k \in K_t$, and $w_t > 0$, by no-arbitrage. For any $\rho_t > 0$, using backward recursion, we construct an optimal solution as follows. For all t and $k \in K_t$, scale c_k by $g_t = 1/\rho_t^{1/(\gamma-1)} > 0$. Then the gradient of the objective function as well as the dual multipliers remain unchanged. To satisfy $c_k - P_k x_{k-} = 0$ at stage T, we scale x_{k-} by

$h_{T-1} = g_T > 0$. For $t = T-1, \ldots, 1$ and $k \in K_t$, to determine the scaling factor h_{t-1} for x_{k-}, it follows from $P_k x_k = w_t c_k$ that $c_k + P_k x_k = (1+w_t)c_k = P_k x_{k-}$ and $h_t P_k x_k = (h_t/g_t) w_t g_t c_k$. To meet the cash balance equation, we require $g_t c_k + h_t P_k x_k = h_{t-1} P_k x_{k-}$ or equivalently, $(1+(h_t/g_t)w_t) g_t c_k = h_{t-1}(1+w_t)c_k$. Solving for h_{t-1} yields $h_{t-1} = (g_t + h_t w_t)/(1+w_t) > 0$. To satisfy the cash balance at the root node, we define the initial endowment by $g_0 c_0 + h_0 P_0 x_0 = (g_0 + h_0 w_0) c_0 > 0$. Consequently, (i) implies (ii).

To show (iii), consider optimization over a subtree with any node $k \in K_t$ as the root node. For initial endowment equal to 1 at k, let U_t denote the optimal expected utility over the subtree. In this notation, we reformulate problem (19) as

$$\max\{\ u_0(c_0) + \sum_{j \in J_0} \pi_j \rho^* u_j(P_j x_0) \mid c_0 + P_0 x_0 = e_0,\ E_0 x_0 \geq 0\ \}, \tag{20}$$

where $\rho^* = (\gamma/\rho_1) U_1 > 0$, for $\gamma \neq 0$, and $\rho^* = 1/\rho_1$, for $\gamma = 0$. Let the optimal multipliers be λ_0 and $\lambda_j = \pi_j \rho^* (P_j x_0)^{(\gamma-1)}$. Then by (ii), state prices $\kappa_j = \lambda_j/\lambda_0$, for $j \in J_0$, are independent of ρ^*, and therefore, independent of T. We apply this observation for all t considering optimization over a subtree with a root node $k \in K_t$. After scaling, the price vector at k becomes P_0. Then employing (i), we conclude (iii).

For (iv), let $x_0 = x^*$ be the optimal portfolio for (20) with $\rho^* \rho_1 = 1$. Then, for any positive ρ^* and ρ_1, the optimal portfolio $x_0 = x^*/(\rho^* \rho_1)^{1/(\gamma-1)}$ is obtained via scaling, similarly as in case (ii) above. Consequently, the optimal portfolio weights at the root are independent of T, and by (i), independent of e_0. We repeat these arguments for any subtree with root node $k \in K_t$, for $t < T$. After scaling, the price vector at k becomes P_0, and we conclude (iv). ■

Remark. Under assumptions of Lemma 3 (ii), consider the time horizon extended beyond T time steps by $n \geq 0$ steps. For a node k at time T, given an endowment equal to 1 for consumption and investment at k, let U_T denote the optimal expected utility over the subtree with root k. If e_k denotes the optimal endowment at k in the extended problem, then the optimal expected utility at k, for $\gamma \neq 0$, is $e_k^\gamma U_T = \rho^*/\gamma\ (e_k)^\gamma$, where $\rho^* = \gamma U_T > 0$. Hence, the extended problem can be solved using the T period problem and upgrading the utility discounting factor at the stage T by the factor ρ^*. Consequently, by Lemma 3 (ii), valuation of options with a maturity of at most T periods, is independent of $n \geq 0$. Similar arguments and conclusions apply to a logarithmic utility as well.

Arbitrage free bounds for option values are uniform applying to all utility functions considered above, and they are independent of the private endowment process. The upper bound $V^+(f)$ and lower bound $V^-(f)$ for the value $V(f)$ is obtained by linear programming as the largest and smallest values of $\sum_k \kappa_k f_k$ such that $\kappa_k \geq 0$ and $\kappa_0 = 1$ satisfy (13)-(14) for some μ_k and $\nu_k \geq 0$.

Equivalently, taking the duals of these linear programs, strong duality theorem implies that $V^+(f)$ is the smallest value $f_0 - c_0$ such that $f_k - c_k \geq 0$, for $k > 0$, among all portfolio strategies with $e_k = 0$, for all k, and $\overline{x} = 0$. To interpret this, we observe that $-c_0$ is the investment expenditure at the root. Hence $V^+(f)$ is f_0 plus the smallest investment needed to super replicate f, i.e., to create cash flow $c_k \geq f_k$, for all $k > 0$. Similarly, $V^-(f)$ is the largest value $f_0 + c_0$ such that $f_k + c_k \geq 0$, for $k > 0$, for portfolio strategies with $e_k = 0$, for all k. Here c_0 is the cash flow created at the root, $-c_k$ is the repayment at node $k > 0$, and $f_k + c_k \geq 0$ requires that f super replicates repayments. Hence, if the price of contingent claim f is less than $V^-(f)$, then buying the claim and employing portfolio strategy determining $V^-(f)$ results in an arbitrage.

Bounds for the option value V in (17), equivalent to those by Harrison and Kreps [10] for perfect markets, are given by

$$\max_f V^-(f) \leq V \leq \max_f V^+(f). \tag{21}$$

An agent is not willing to pay a price above the upper limit in (21), because a smaller investment would super replicate the option cash flow. On the other hand, if the option price is below the lower limit in (21), an arbitrage opportunity is created for the agent. If F is a convex set, then the left side of (21) is a convex-concave saddle point problem, which can be solved using the method of Kallio and Ruszczyński [16]. The right side is an optimization problem with a pseudo concave objective function. This problem can be solved, for instance, using Minos; Saunders and Murtagh [24].

References

[1] Aseff, J.G. and M.S. Santos (2005) Stock options and managerial optimal contracts. *Economic Theory* 26: 813–837.

[2] Bettis, J.C., Bizjak, J.M. and M.L. Lemmon (2005) Exercise behavior, valuation and the incentive effects of employee stock options. *Journal of Financial Economics* 76: 445–470.

[3] Campbell, J.Y. and L.M. Viceira (2002) *Strategic asset allocation: portfolio choice for long-term investors.* Clarendon Lectures in Economics, Oxford University Press.

[4] Carpenter, J.N. (1998) The exercise and valuation of executive stock options. *Journal of Financial Economics* 48: 127–158.

[5] Davis, S.J., Kubler, F. and P. Willen (2006) Borrowing costs and the demand for equity over the life cycle. *Review of Economics and Statistics* 88: 348–362.

[6] Detemple, J. and J. Sundaresan (1999) Nontraded asset valuation with portfolio constraints: a binomial approach. *Review of Financial Studies* 12: 835–872.

[7] Fourer, R., D.M. Gay, and B.W. Kernighan (2003), *AMPL, A Modeling Language for Mathematical Programming.* Nelson Thomson Learning, Toronto, Canada.

[8] Grossman, S.J. and O.D. Hart (1983) An analysis of the principal-agent problem. *Econometrica* 51: 7–45.

[9] Hall, B.J. and K.J. Murphy (2002) Stock options for undiversified executives. *Journal of Accounting and Economics* 33: 3–42.

[10] Harrison, J.M. and D.M. Kreps (1979) Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* 20: 381–408.

[11] Henderson, V. (2005) Explicit solutions to an optimal portfolio choice problem with stochastic income. *Journal of Economic Dynamics and Control* 29: 1237–1266.

[12] Ikäheimo, S., N. Kuosa, and V. Puttonen (2006) The 'true and fair view' of executive stock option valuation. *European Accounting Review* 15: 351–366.

[13] Ingersoll, J.E. (2006) The subjective and objective evaluation of incentive stock options. *Journal of Business* 79: 453–487.

[14] Kahl, M., Liu, J. and F. A. Longstaff (2003) Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? *Journal of Financial Economics* 67: 385–410.

[15] Kallio, M. (2005) Real Option Valuation under Imperfect Markets. Helsinki School of Economics.

[16] M. Kallio and A. Ruszczyński (1994) Perturbation Methods for Saddle Point Computation. Working Paper WP-94-38, IIASA, Laxenburg.

[17] Kallio, M. and W.T. Ziemba (2007) Using Tucker's theorem of the alternative to simplify, review and expand discrete arbitrage theory. *Journal of Banking and Finance* (article in press).

[18] Koo, H.K. (1998) Consumption and portfolio selection with labor income: a continuous time approach. *Mathematical Finance* 8: 49–65.

[19] Koo, H.K. (1999) Consumption and portfolio selection with labor income: a discrete-time approach. *Mathematical Methods of Operations Research* 50: 219–243.

[20] Leone, A.J., Wu, J.S. and J.L. Zimmermann (2006) Asymmetric sensitivity of CEO cash compensation to stock returns. *Journal of Accounting and Economics* 42: 167–192.

[21] Mangasarian, O. (1971) *Nonlinear Programming*, McGraw-Hill Book Company.

[22] Merton, R.C. (1971) Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory* 3: 373–413.

[23] Merton, R.C. (1973) An inter-temporal capital asset pricing model. *Econometrica* 41: 867–887.

[24] Murtagh, B., and M. Saunders (1978) Large-Scale Linearly Constrained Optimization. *Mathematical Programming*, 14: 41–72.

[25] Pratt, J.W. (1964) Risk aversion in the small and in the large. *Econometrica* 32: 122–136.

[26] Wets, R. and W. Ziemba (1999) Stochastic Programming. State of the Art, 1998. *Annals of Operations Research* 85.