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for Interactive Nonlinear Multiobjective Optimization

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Abstract

When solving multiobjective optimization problems with interactive methods, a decision maker (DM) directs the search for the most preferred Pareto optimal solution with his/her preferences. We propose a tool that can be used to support the DM. With this tool, the DM can conveniently learn about local trade-offs between the conflicting objectives and judge whether they are worthwhile. The tool is based on an idea where the DM is able to vary a selected Pareto optimal objective vector. The varied vector is treated as a reference point which is then projected to the tangent hyperplane of the Pareto optimal set at the Pareto optimal solution selected. This information can be used to reflect what kind of Pareto optimal solutions and trade-offs are available in a local neighborhood of the solution. This tool is especially useful when trade-off analysis must be carried out fast and without increasing computation workload. We also propose a applicability study method for the trade-off analysis tool proposed and demonstrate the usage of the ideas through an academic example problem.

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1 Introduction

When solving multiobjective optimization problems, we must optimize several conflicting objective functions simultaneously. Because of the conflicting nature of the objectives, we can identify compromises, so-called Pareto optimal solutions, where we cannot improve any objective without impairing at least one of the others. The purpose of multiobjective optimization methods is to offer support and ways to find the best compromise solution. In this, a decision maker (DM) and his/her preference information play an important role. By a DM we mean a person who is an expert in the domain of the problem considered and who typically is responsible for the final solution.

Multiobjective optimization methods can be classified in many ways (see e.g. [6]). A widely used class of methods is interactive methods where the DM iteratively directs the solution procedure by indicating his/her preferences related to solution candidates shown. The method then utilizes the given preference information and tries to produce new solution candidates which are more satisfying for the DM. The iteration continues until the DM is satisfied or likes to stop the solution process. The key feature of interactive methods is that during the solution process the DM is able to learn about the underlying problem as well as his/her own preferences.

So far, many interactive methods have been proposed in the literature (see e.g. [6]). They differ from each other, for example, by the type of preference information utilized. It is important that the DM is able to specify the kind of information (s)he is asked to and that the concepts used are familiar to him/her. Reference point based interactive methods are popular because a reference point has a natural meaning for the DM. It consists of desirable objective function values, so-called aspiration levels (for each objective). After the DM has specified the reference point, the feasible Pareto optimal solution is found that best corresponds to it. If the DM is not satisfied, (s)he can specify another reference point. However, these methods are sometimes criticized for the fact that they do not provide support for the DM how to change the reference point in order to get more preferred solutions.

In this paper, we propose a trade-off analysis tool that can offer for the DM a way to analyze solution candidates. This tool is best suited for local analysis of solutions where the DM is interested in studying whether it is worthwhile to search for a better solution in a neighborhood of some solution. This study is strongly motivated by the experiences given by several DMs who have used the interactive NIMBUS method (see e.g.[6, 7]). NIMBUS is a classification based method and it is shown in [7] how it is closely related to reference point based methods. The ideas proposed are directed to users of both classification and reference point based methods. The motivation here is that real DMs in certain cases miss additional local trade-off information so that they could get to know how values of objectives are changing, in other words, in which directions to direct the solution process so that they could avoid trial-and-error, that is, specify some preference information so that more preferred solutions will be generated. Especially, in the case where the underlying problem is computationally demanding, it does not necessarily make sense to compute additional solutions around some solution just to grasp an idea of what kind of trade-offs there are available because such an approach is too time-

consuming. Instead, the local trade-off analysis is computationally very inexpensive because it utilizes information that is already available as a byproduct of previous optimizations.

Examples of interactive methods that utilize objective trade-off information include ISWT [1], SPOT [9], GRIST [12], STOM [8] and IMOOP [10]. By objective trade-off information we mean that information is based on the properties of the underlying problem. In the ISWT, SPOT, and GRIST methods objective trade-offs are in crucial role, that is, these methods use trade-off information explicitly when they determine Pareto optimal solutions based on the DM's preferences. In the STOM method trade-offs can be also used explicitly but the DM is allowed to make this decision, that is, objective trade-offs are not necessarily taken account when determining Pareto optimal solutions. The IMOOP method gives us an example of a method where objective trade-off information is offered for the DM just as additional information that can be used in analysis purposes but it is not explicitly used when Pareto optimal solutions are determined. In other words in IMOOP the DM is free to use shown trade-off information as (s)he likes. See [2] for more details about different kind of interactive multiobjective methods that utilize objective trade-offs.

The trade-off analysis tool to be presented in this paper is a bit similar to the automatic trade-off concept in the interactive STOM method where the DM classifies objectives at each iteration into three classes (improved, relaxed, or accepted as they are). In STOM, a central idea is that the DM only gives desirable amounts of change for the objectives to be improved and objective trade-off information is used to estimate how much relaxation is needed in the others to obtain the desired improvements. The aim is to decrease the cognitive burden set on the DM by asking less preference information.

The main ideological difference between STOM and our approach is that we aim at providing the DM support for specifying the classification, that is, which objectives to improve or to relax. In our approach, the DM is assumed to determine desires of how to change objective values to get a more preferred solution. However, instead of using this preference information to generate new Pareto optimal solutions, we use trade-off analysis to see whether desired improvements are possible. In other words, we use a linear approximation of the solution set to reflect local trade-offs near some solution of a nonlinear problem. Because trade-off analysis is in our approach computationally inexpensive, we can save in computational cost. Furthermore, it may take time to generate the new solution. With our approach, the DM can get confirmation whether it really makes sense to go to the direction specified and wait for the new solution to be produced. When the DM has found an interesting direction the computation can be executed to produce a Pareto optimal solution.

The authors have already shortly introduced initial ideas behind the trade-off analysis tool in [3]. However, in this paper these ideas are developed further and discussed in more detail. In addition a supplementary applicability study method is included.

In what follows, in Section 2 we briefly present concepts and notation. The trade-off analysis tool is introduced in Section 3. In Section 4, we propose an approach how to study an applicability of the trade-off analysis. The usage of the method

proposed is demonstrated through an example in Section 5. Finally, we discuss the potential of the tool in Section 6 and conclude in Section 7.

2 On multiobjective optimization and interactive methods

We consider nonlinear multiobjective optimization problems in the form

$$
\begin{aligned}
\text{minimize} \quad & \{f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x})\} \\
\text{subject to} \quad & \boldsymbol{x} \in S
\end{aligned}
\tag{1}
$$

with k conflicting twice differentiable objective functions f_i which are minimized subject to *decision vector* $\boldsymbol{x}$ belonging to a *feasible set* S. For each $\boldsymbol{x} \in S$ we can use a mapping $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^k$ to form a *feasible objective vector* $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_k(\boldsymbol{x}))^T$ belonging to an *objective space* $\mathbb{R}^k$. The image $\boldsymbol{f}(S) \subseteq \mathbb{R}^k$ of the feasible set is called a *set of feasible objective vectors*.

For the DM, potentially interesting solutions of problem (1) can by identified by using the concept of Pareto optimality. A decision vector $\dot{\boldsymbol{x}} \in S$ and the corresponding objective vector $\boldsymbol{f}(\dot{\boldsymbol{x}})$ are *Pareto optimal* if there exists no other decision vector $\boldsymbol{x} \in S$ such that $f_i(\boldsymbol{x}) \leq f_i(\dot{\boldsymbol{x}})$ for all $i = 1, \ldots, k$ and at least one of the inequalities is strict. In what follows, we refer to a Pareto optimal decision vector and the corresponding objective vector by the term *Pareto optimal solution*. If a solution is Pareto optimal, we say that it is not *dominated* by any other feasible solution, that is, it is nondominated. In this text, when it is said that an objective vector is dominated or nondominated we mean the dominance in the sense of Pareto optimality. In objective space Pareto optimal objective vectors can be identified by using positive orthant cone $\mathbb{R}_+^k =: \{\boldsymbol{z} \in \mathbb{R}^k : z_i \geq 0, \text{ for } i = 1, \ldots, k\}$ as a *dominance structure*. If $\dot{\boldsymbol{z}}$ is Pareto optimal then $(\dot{\boldsymbol{z}} - \mathbb{R}_+^k) \cap (\boldsymbol{f}(S) \setminus \{\dot{\boldsymbol{z}}\}) = \emptyset$. We use a term *Pareto surface* for the set $P = \boldsymbol{f}(E)$, where $E \subseteq S$ is the set of Pareto optimal solutions of problem (1). A decision vector $\dot{\boldsymbol{x}} \in S$ is *weakly Pareto optimal* if there does not exist another decision vector $\boldsymbol{x} \in S$ such that $f_i(\boldsymbol{x}) < f_i(\dot{\boldsymbol{x}})$ for all $i = 1, \ldots, k$. The set of Pareto optimal solutions is always a subset of the set of weakly Pareto optimal solutions.

A general interactive multiobjective optimization method can be outlined as follows:

1. Find an initial Pareto optimal solution.
2. Interact with the DM.
3. Obtain a Pareto optimal solution. If the new solution or some of the previous solutions is acceptable to the DM, stop. Otherwise, go to step 2.

The main advantage of this kind of interactive methods is that the DM is allowed to guide the solution procedure to areas of the feasible set where the most interesting Pareto optimal solutions are located.

Here we concentrate on reference point and classification based methods. To be more specific, we assume that an achievement scalarizing problem [11] is used

in steps 1 and 3 to produce (weakly) Pareto optimal solutions for problem (1). In other words, we solve the problem

$$
\begin{array}{ll}
\text{minimize} & \alpha \\
\text{subject to} & w_i(f_i(\boldsymbol{x}) - \bar{\boldsymbol{z}}) \leq \alpha, \text{ for } i = 1, \ldots, k \\
& \boldsymbol{x} \in S, \alpha \in \mathbb{R}.
\end{array} \tag{2}
$$

For brevity of presentation, we discuss a scalarizing formulation that generates weakly Pareto optimal solutions. Weak Pareto optimality can be avoided, for instance, by considering an augmented version (see e.g. [6]). In step 2 of a reference point based interactive procedure, the DM sets aspiration levels $\bar{z}_i$ to indicate desirable levels of the objective functions f_i, for $i = 1, \ldots, k$. These values can be used to form a *reference point* $\bar{\boldsymbol{z}} = (\bar{z}_1, \ldots, \bar{z}_k)^T$. The *scaling coefficients* $w_i > 0$ can be used to determine how the given reference point $\bar{\boldsymbol{z}}$ is projected to the set of Pareto optimal solutions. One possible scaling is $w_i = 1/(z_i^{\text{up}} - z_i^{\text{low}})$, where z_i^{up} and z_i^{low} are approximated (computed or given by the DM) high and low value for the objective function f_i attainable in the Pareto optimal set, respectively. In what follows, when we say that $\dot{\boldsymbol{x}}$ is an optimal solution of (2) we implicitly refer to solution $(\dot{\boldsymbol{x}}, \dot{\alpha})$.

3 Interactive trade-off analysis tool

Assuming the problem is properly defined and the DM is rational, the DM considers Pareto optimal solutions in multiobjective optimization methods. Because of the definition of Pareto optimality, moving from some Pareto optimal solution to another one always necessitates some trade-off in objective function values. When using an interactive method, the DM might be interested in knowing what kind of trade-off takes place if some particular Pareto optimal solution is altered, that is, what kind of Pareto optimal solutions exists in a neighborhood of the Pareto optimal solution in investigation. From the practical point of view, it is not always necessarily efficient or purposeful to carry out accurate computations to reflect what is happening in the set of Pareto optimal solutions. In such a case an approximation of the Pareto surface can be used to study interconnections between the objectives.

3.1 Linear approximation

Let us assume that a Pareto optimal solution $\dot{\boldsymbol{x}}$ for problem (1) is produced by solving problem (2) and the DM wants to study the trade-offs in a local neighborhood of $\dot{\boldsymbol{z}} = \boldsymbol{f}(\dot{\boldsymbol{x}})$ in order to determine in which direction to move from this solution to get a more satisfactory solution. When available, a tangent hyperplane $T(\dot{\boldsymbol{z}})$ for the Pareto surface P at $\dot{\boldsymbol{z}}$ can be used to reflect trade-off information. In other words, we produce a linear approximation for the nonlinear (possibly also nonconvex) Pareto surface. In what follows, we refer to $T(\dot{\boldsymbol{z}})$ with a term *trade-off plane*. The trade-off interpretation of Karush-Kuhn-Tucker (KKT) multipliers related to a solution of problem (2) can be used to obtain this trade-off plane [14], [15].

Theorem 3.1 (Partial trade-off using KKT multipliers)**.** *Let $\dot{\boldsymbol{x}}$ be a solution of (2) for some $\boldsymbol{w}$ and $\bar{\boldsymbol{z}}$, with optimal KKT multipliers λ_i related to constraints $w_i(f_i(\boldsymbol{x}) -$*

$\bar{z}_i) \leq \alpha$, for $i = 1, \ldots, k$. *Let us assume that 1. second-order KKT sufficiency conditions are satisfied at $\dot{x}$, 2. at $\dot{x}$, gradients of active constraints are linearly independent and 3. all active constraints have strictly positive KKT multipliers at $\dot{x}$. Then partial trade-off rate involving objectives f_1 and f_i at $\dot{x}$ is $\partial f_1(\dot{x})/\partial f_i = -(\lambda_i w_i)/(\lambda_1 w_1)$, for $i = 2, \ldots, k$.*

Partial trade-off rate information can be interpreted as a linear approximation for a relative change in the value of f_1 when the value of f_i ($i = 2, \ldots, k$) is changing by one unit and at the same time all the other objectives f_j ($j = 2, \ldots, k$, $j \neq i$) remain at their current levels. This information can be presented in the form of *partial trade-off vectors* $\boldsymbol{t}_{1i}(\dot{x}) \in \mathbb{R}^k$, where the first component is $-(\lambda_i w_i)/(\lambda_1 w_1)$, i:th component is 1, and the other components ($j = 2, \ldots, k$, $j \neq i$) are zero. We can directly conclude that vector $\boldsymbol{n} = -(\lambda_1 w_1, \lambda_2 w_2, \ldots, \lambda_k w_k)^T$ is orthogonal to $\boldsymbol{t}_{1i}(\dot{x})$, for all $i = 2, \ldots, k$. In other words, we can express a normal vector in an explicit form whenever we are able to compute partial trade-off vectors (see e.g. [12] and [13]).

The normal vector $\boldsymbol{n}$ for the Pareto surface P at some $\dot{z} \in P$ can be used to characterize objective vectors $\boldsymbol{z} \in T(\dot{z}) \subset \mathbb{R}^k$ in the neighbourhood of $\dot{z}$ in the form $\boldsymbol{n}^T(\boldsymbol{z} - \dot{z}) = 0$. Because the plane $T(\dot{z})$ is giving only a linear approximation for the Pareto surface, we must point out that in the nonlinear case a vector $\boldsymbol{z} \in T(\dot{z})$ is most likely not Pareto optimal (when $\boldsymbol{z} \neq \dot{z}$), and it might be even infeasible. However, we can use $T(\dot{z})$ to roughly reflect what kind of Pareto optimal objective vectors are available in a local neighborhood of $\dot{z}$.

In this paper, we restrict our consideration to Pareto optimal solutions of problem (1) where it is possible to compute the trade-off plane $T(\dot{z})$ using KKT multipliers (see e.g. [5] how the assumptions in Theorem 3.1 can be verified). Based on this information, the DM can judge whether more preferred solutions could be located in a neighborhood of the Pareto optimal solution $\dot{z}$, in other words, in which directions to look for better solutions by executing steps of the actual interactive procedure (without approximation). Next we propose a trade-off analysis tool and discuss it in some more detail. We assume that KKT multipliers are available as byproducts after solving problem (2).

3.2 Trade-off analysis tool

Let us assume that the DM wants to study the trade-off behavior in a local neighborhood of a Pareto optimal solution $\dot{z}$. The DM is assumed to determine desires of how to change objective values to get a more preferred solution. However, instead of using this preference information to generate new Pareto optimal solutions, we use trade-off analysis to study whether desired improvements are possible. This analysis is computationally inexpensive because it is based on information obtained from KKT multipliers related to the solution $\dot{z}$. Whenever we use appropriate numerical solver to solve problem (2) constraint related KKT multipliers can be obtained as a byproduct. Therefore, with the trade-off tool, the DM can conveniently and fast get confirmation whether it really makes sense to go to the direction specified. Alternatively, if the solution is far from what was expected, the DM may alter one's preferences.

In what follows, a vector $\boldsymbol{d} \in \mathbb{R}^k$ (expressed implicitly by the DM) is used to indicate desired changes in the objective function values at $\dot{\boldsymbol{z}}$. This determines a reference point $\bar{\boldsymbol{z}} = (\dot{\boldsymbol{z}} + \boldsymbol{d})$ which can be used in a linear achievement scalarizing problem

$$
\begin{aligned}
\text{minimize} \quad & \alpha \\
\text{subject to} \quad & w_i(z_i - (\dot{z}_i + d_i)) \leq \alpha, \text{ for } i = 1, \ldots, k \\
& \boldsymbol{n}^T(\boldsymbol{z} - \dot{\boldsymbol{z}}) = 0 \\
& \boldsymbol{z} \in \mathbb{R}^k, \alpha \in \mathbb{R}.
\end{aligned} \tag{3}
$$

In other words, the Pareto surface is temporarily replaced by a linear approximation at $\dot{\boldsymbol{z}}$ (see Figure 1) and the solution which is in some sense closest to the reference point locating on this trade-off plane (in Figure 1 this solution is denoted by $\tilde{\boldsymbol{z}}$) is found and shown to the DM. We call this solution an *approximated Pareto optimal solution*. This gives some rough information for the DM about the feasibility of the desires expressed in the reference point. Because the approximated solution corresponding to the new reference point is obtained fast, the DM can judge in real-time whether (s)he would like to get the actual Pareto optimal solution, that is, solve problem (2) corresponding to the reference point $\bar{\boldsymbol{z}}$.

It is preferable, even thought not necessary, that the variation $\boldsymbol{d}$ is set in such a way that it obeys the concept of Pareto optimality, that is, if value of some objective is improved then the value of at least one objective should be degraded. This is an important issue because if the preferences are set according to the concept of Pareto optimality it is more likely that the DM can experience that the method is able to produce Pareto optimal solutions which corresponds to preferences expressed (feeling of control). Problem (3) has a unique solution except if the normal vector $\boldsymbol{n}$ is parallel to some of the coordinate axis in the objective space $\mathbb{R}^k$.

For our purposes, it is enough to use information that the projection direction in problem (2) is determined by a vector $\boldsymbol{w}^{-1} = (1/w_1, \ldots, 1/w_k)$, where $w_i > 0$ (i=1,...,k) are the scaling coefficients (we do not have to pay special attention to weak Pareto optimality). Thus, instead of solving the linear problem (3), we can directly compute where the projection direction vector $(\dot{\boldsymbol{z}} + \boldsymbol{d} + t\boldsymbol{w}^{-1})$ and trade-off plane $\boldsymbol{n}^T(\boldsymbol{z} - \dot{\boldsymbol{z}}) = 0$ are intersecting. This means solving for t (to be denoted by $\tilde{t}$) in equation $\boldsymbol{n}^T((\dot{\boldsymbol{z}} + \boldsymbol{d} + t\boldsymbol{w}^{-1}) - \dot{\boldsymbol{z}}) = 0$ which we can write in the form $\tilde{t} = -(\boldsymbol{n}^T\boldsymbol{d})/(\boldsymbol{n}^T\boldsymbol{w}^{-1})$ (we do not need to consider case $\boldsymbol{n}^T\boldsymbol{w}^{-1} = 0$ because $w_i^{-1} > 0$, $n_i \leq 0$, and $\boldsymbol{n} \neq \boldsymbol{0}$). Then, the approximated Pareto optimal objective vector obtained using the preference information given by the DM is $\tilde{\boldsymbol{z}} = (\dot{\boldsymbol{z}} + \boldsymbol{d}) + \tilde{t}\boldsymbol{w}^{-1} = \bar{\boldsymbol{z}} + \tilde{t}\boldsymbol{w}^{-1}$.

In practical method framework it is convenient to assumed that $\boldsymbol{z}^{\text{low}} \leq \bar{\boldsymbol{z}} \leq \boldsymbol{z}^{\text{up}}$ and $\boldsymbol{z}^{\text{low}} \leq \tilde{\boldsymbol{z}} \leq \boldsymbol{z}^{\text{up}}$. However, in the above procedure it might happen that $\tilde{z}_i \geq z_i^{\text{up}}$, for more than one $i = 1, \ldots, k$, even though $\boldsymbol{z}^{\text{low}} \leq \bar{\boldsymbol{z}} \leq \boldsymbol{z}^{\text{up}}$ would hold. This special case can be also taken account but for the sake of clarity we omit such a details in this paper.

Figures 1 and 2 depict an imaginary situation where the trade-off analysis is carried out at some Pareto optimal solution $\dot{\boldsymbol{z}} \in P$. It must be pointed out that even thought in Figures 1 and 2 we consider the case of only two objectives, the trade-off analysis tool is best suited for problems having three or more objectives. Naturally, the method can also be used without any restrictions for biobjective
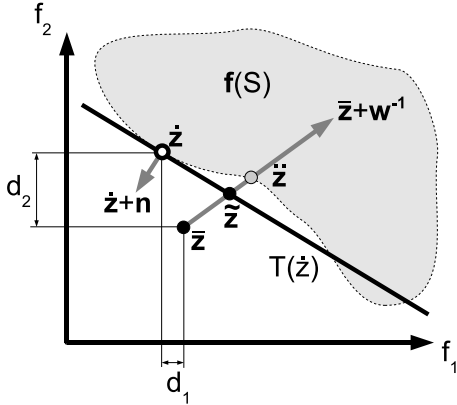
Figure 1: Approximation of the Pareto surface

Figure 2: Varying the selected solution

problems but it is not that useful.

In Figure 1, we demonstrate the trade-off tool with a simple problem, where the DM considers solution $\dot{z}$ and is interested in improving f_2 value by allowing f_1 to degrade as indicated by the reference point $\bar{z} = (\dot{z} + \boldsymbol{d})$. According to the trade-off analysis, the approximated solution reflecting $\bar{z}$ is $\tilde{z} \in T(\dot{z})$. Vector $\ddot{z}$ is the Pareto optimal objective vector which would have been obtained by solving problem (2). The Pareto optimal vector corresponding to $\bar{z}$ is computed only if the approximated Pareto optimal objective vector $\tilde{z}$ is promising for the DM. From $\tilde{z}$, the DM gets fast some understanding about the feasibility of the desires specified without solving (2) at all. It is clear that in a general nonlinear case, too large variation $\boldsymbol{d}$ may lead to a poor approximation. In practice, the DM can be informed about an appropriate maximal variation (see the next section).

Figure 2 gives an example from a practical point of view by suggesting a user interface for the trade-off analysis tool described above. The DM has selected for trade-off analysis a Pareto optimal solution $\dot{z}$ for which objective function values are visualized using a bar chart. The triangle marker is a reference point $\bar{z} = (\bar{z} + \boldsymbol{d})$ determined by the DM. The circle marker indicates the approximated Pareto optimal objective vector $\tilde{z} \in T(\dot{z})$ obtained using trade-off analysis.

When the reference point $\bar{z}$ is altered the approximated vector $\tilde{z}$ can be updated very fast accordingly and the DM can get in real-time an idea of what kind of trade-offs are available in a local neighborhood of the selected Pareto optimal solution. Furthermore, the DM can learn how his/her preferences corresponds to Pareto optimal solutions available near $\dot{z}$.

4 Applicability study

In the previous section we did introduce the trade-off analysis tool which can be used to study what kind of Pareto optimal solutions and trade-offs there are available around some Pareto optimal solution. The strength of this approach is that all computationally significant technical information needed to setup the analysis environment can be obtained as a byproduct when the Pareto optimal solution is
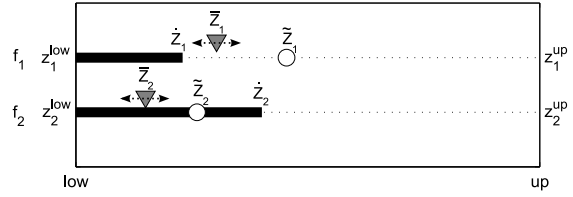
produced as a solution of (2) by using an appropriate single objective optimization algorithm able to produce KKT multipliers.

Even thought the proposed tool sounds appealing, especially from the computational point of view, it must be pointed out that the analysis is based on a linear approximation of the Pareto surface. It is clear that for highly nonlinear Pareto surfaces the linear approximation may be reliable only in a very (possibly insignificantly) small neighborhood of the Pareto optimal solution selected for the analysis. Therefore, it is important to have some indicators which can be used to say something about the applicability of the approximated Pareto optimal solutions. The aim of the trade-off analysis tool is to offer support for decision making and learning. We do not want to mislead or distract the DM and this gives us a motivation to study applicability issues in more detail.

In a general case, it is not easy to study how well the linear trade-off plane at some Pareto optimal solution is approximating the nonlinear (possibly nonconvex) Pareto surface. This is even more difficult if we are not willing to make any additional (with respect to Theorem 3.1) technical assumptions related to problem properties. Without restricting the applicability of the trade-off analysis tool, in what follows, we propose a very general approach which allows us to say at least something about how realistic the approximation is. It must be emphasized that an applicability study approach to be presented is just an addition to the trade-off tool idea introduced in the previous section. When the DM is just interested in knowing trade-off rates between objectives the trade-off tool can be useful even thought the approximated Pareto optimal solution is not realistic. However, if the DM is wants to use trade-off tool explicitly to set reference points then it is of course interesting to know whether the approximated Pareto optimal solution is realistic.

In our approach, we propose an applicability study which is solely based on the Pareto optimal solutions computed during an interactive procedure. In other words, we do applicability study without producing any additional information. We do not pay any attention to the technical properties of the underlying problem (1) but use only the information that a Pareto optimal objective vector cannot be dominated by any other Pareto optimal objective vector. Roughly speaking, when in the trade-off analysis the DM determines a reference point by varying the values of some Pareto optimal objective vector, we can study whether the corresponding approximated Pareto optimal objective vector belonging to the trade-off plane is dominated by or dominating any of the known Pareto optimal objective vectors. In other words, we can determine whether the approximation is realistic with respect to all Pareto optimal solutions computed. Therefore, the applicability study can be used to indicate for the DM unrealistic approximations. In this framework more accurate control is obtained when the number of available Pareto optimal points increases. However, it must be emphasized that our aim is not to restrict the trade-off analysis but just indicate for the DM that the variation proposed to the Pareto optimal solution under investigation might be too rough and therefore it would be probably better to consider smaller variation or do the analysis based on some other Pareto optimal solution.

The idea of applicability study is illustrated in Figure 3 where the analysis is carried out at a Pareto optimal objective vector $\dot{z}^0$. We assume that at the moment
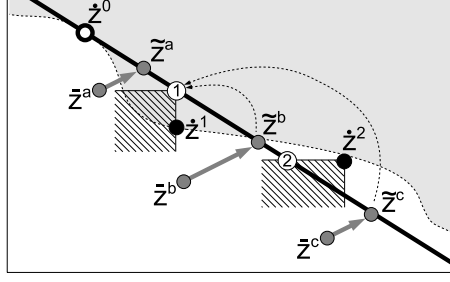
Figure 3: Applicability study of the approximation

when the trade-off analysis takes place, only Pareto optimal objective vectors $\dot{z}^0$, $\dot{z}^1$, and $\dot{z}^2$ are explicitly known. The straight line going through $\dot{z}^0$ is depicting the trade-off plane $T(\dot{z}^0)$. Figure 3 contains three cases where the DM has independently determined $\bar{z}^a$, $\bar{z}^b$, or $\bar{z}^c$ as a reference point.

In Figure 3, the reference point $\bar{z}^a$ produces an approximated Pareto optimal objective vector $\tilde{z}^a$. Based on the Pareto optimal objective vectors $\dot{z}^1$ and $\dot{z}^2$ this approximation is realistic, that is, $\tilde{z}^a$ is neither dominated by nor dominating the other known Pareto optimal objective vectors. A similar conclusion can be drawn in the case of any objective vector $(\dot{z}^0 + t(\tilde{z}^a - \dot{z}^0)) \in T(\dot{z}^0)$, for $t \in [0, 1]$. In other words, any of the objective vectors $(z^0 + t(\tilde{z}^a - \dot{z}^0))$ can be Pareto optimal with respect to $\dot{z}^1$ and $\dot{z}^2$ (note that explicit form of the Pareto surface is not known). We say that a Pareto optimal objective vector is *active* when it dominates or is dominated by vector $(z^0 + t(\tilde{z}^a - \dot{z}^0))$, for some $t \in [0, 1]$.

Let us now assume that the DM has determined $\bar{z}^b$ or $\bar{z}^c$ as a reference point. Even thought both the resulting approximated Pareto optimal objective vectors $\tilde{z}^b$ and $\tilde{z}^c$, respectively, can be Pareto optimal (neither dominated nor dominating) the problem is that the objective vectors $(\dot{z}^0 + t(\tilde{z}^b - z^0)) \in T(\dot{z}^0)$ and $(\dot{z}^0 + t(\tilde{z}^c - \dot{z}^0)) \in T(\dot{z}^0)$ both violate the domination concept for some $t \in [0, 1]$ (in Figure 3 circled numbers 1 and 2 indicate the position where the corresponding Pareto optimal objective vector becomes active). Therefore, if the DM proposes $\bar{z}^b$ or $\bar{z}^c$ as a reference point, the method can inform the DM about the Pareto optimal solution which becomes first active when we start to move from the Pareto optimal $\dot{z}^0$ toward to the approximated Pareto optimal objective vector $\tilde{z}^b$ or $\tilde{z}^c$. In other words, we see from Figure 3 that the DM might be interested in performing trade-off analysis in a local neighborhood of $\dot{z}^1$ instead of $\dot{z}^0$ if (s)he proposes variation to $\dot{z}^0$ which leads to the reference point $\bar{z}^b$ or $\bar{z}^c$. In this context, the reference point $\bar{z}^c$ does not imply that the method suggests Pareto optimal objective vector $\dot{z}^2$ for the DM because it is assumed that the DM is interested in doing local analysis around Pareto optimal $\dot{z}^0$ and therefore the first active Pareto optimal objective vector $\dot{z}^1$ is suggested instead.

In must be emphasized that in this applicability study it is assumed that the DM is especially interested about the solutions which have similar trade-off rates between objectives as induced by the reference point. In the terms of Figure 3 the trade-off rates related to the approximated Pareto optimal objective vectors $(\dot{z}^0 + t(\tilde{z}^c - \dot{z}^0))$ are corresponding to reference points $\dot{z}^0 + t(\bar{z}^c - \dot{z}^0)$, for $t \in [0, 1]$. In other words,

if approximated Pareto optimal objective vector $\tilde{z}^c$ corresponding to reference point $\tilde{z}^c$ is feasible to the DM but not realistic then it is assumed that the DM is also interested in considering approximated Pareto optimal vectors $(\dot{z}^0 + t(\tilde{z}^c - \dot{z}^0))$, for $t \in [0, 1]$, because similar trade-off rates are assigned to them.

In an interactive solution procedure, usually some set of Pareto optimal solutions is generated based on the DM's preference information. Let us denote by $P' \subset P$ a set of Pareto optimal objective vectors produced in the interactive method according to the preferences of the DM. We assume that the DM wants to do trade-off analysis related to the Pareto optimal solution $\dot{z} \in P'$ by proposing a variation d_i for some values $\dot{z}_i$, for $i = 1, \ldots, k$. This produces a reference point $\bar{z} = (\dot{z} + \boldsymbol{d})$. When the Pareto optimal solution $\dot{z}$ is varied by $\boldsymbol{d}$ it might happen that for some $t \in [0, 1]$ approximated Pareto optimal solution $\dot{z} + t(\tilde{z} - \dot{z})$ becomes dominated by or starts to dominate some of the Pareto optimal solutions in set $P'' = P' \setminus \{\dot{z}\}$. If this happens, we know that the approximated Pareto optimal objective vector is not necessarily realistic. If no such $t \in [0, 1]$ exists, then the obtained linear approximation can be considered to be applicable. However, we do not actually have an idea how far away the approximated Pareto optimal objective vector is from the Pareto surface.

Algorithm 1 below can be used to determine whether some approximated Pareto optimal objective vector is realistic with respect to the set P''. The idea is to find the minimal $t \in [0, 1]$ in such a way that either $\dot{z} + t(\tilde{z} - \dot{z})$ enters the set $\boldsymbol{z} - \mathbb{R}_+^k$ or $\boldsymbol{z}$ enters the set $\dot{z} + t(\tilde{z} - \dot{z}) - \mathbb{R}_+^k$, where $\boldsymbol{z} \in P''$.

Algorithm 1 Applicability study of approximated Pareto optimal objective vector

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Require: $\dot{z} \in P'$ and $\tilde{z} \in T(\dot{z})$
  $t_{min} \leftarrow M > 1$
  $\boldsymbol{z}_{min} \leftarrow \dot{z}$
  for all $\boldsymbol{z} \in P''$ **do**
    $t_1 \leftarrow$ minimize t subject to $(\tilde{z} - \dot{z})t \leq (\dot{z} - \boldsymbol{z})$ and $0 \leq t \leq 1$
    if $t_1 \leq t_{min}$ **then**
      $t_{min} \leftarrow t_1$
      $\boldsymbol{z}_{min} \leftarrow \boldsymbol{z}$
    else
      $t_2 \leftarrow$ minimize t subject to $(\tilde{z} - \dot{z})t \geq (\dot{z} - \boldsymbol{z})$ and $0 \leq t \leq 1$
      if $t_2 \leq t_{min}$ **then**
        $t_{min} = t_2$
        $\boldsymbol{z}_{min} \leftarrow \boldsymbol{z}$
      end if
    end if
  end for

If we have $t_{min} = M$ at the end, then the approximated Pareto optimal objective vector $\tilde{z}$ can be Pareto optimal with respect to P''. The first minimization problem reveals whether $\dot{z} + t(\tilde{z} - \dot{z})$ dominates any Pareto optimal objective vectors in P'' for some $t \in [0, 1]$ and the second one whether $\dot{z} + t(\tilde{z} - \dot{z})$ is dominated by any Pareto optimal objective vectors in P''. If at the end $0 \leq t \leq 1$ then $\boldsymbol{z}_{min}$ refers to the Pareto optimal objective vector which becomes first active when we move from

$\dot{z}$ to $\tilde{z}$.

We assume that $\dot{z} \neq z$ holds for all pairs $\dot{z}, z \in P'$. However, when $k > 2$ it might happen that t_{min} value is assigned to more than one Pareto optimal solution. In such a case we can choose in some sense closest one of these solutions. Weak Pareto optimality might also necessitate a special attention especially in the case when $t_{min} = 0$. However, we do not go into details with these issues because the emphasis is on the description of a general method framework.

It is clear that when we have only very few Pareto optimal points available (or if we have a large number of objectives), the applicability study method proposed does not necessarily help that much to evaluate whether approximated Pareto optimal solutions obtained in the trade-off analysis are realistic or not (this is due the fact that considered objective vectors are not dominating each other). In other words, the DM can determine very large variations to the Pareto optimal objective vector under investigation without the method being able to say anything about the applicability of the approximation obtained. However, the applicability study presented in this section can be regarded as an addition to the initial trade-off analysis tool idea rather than a robust applicability indicator. The idea is just to take advantage of all information available, that is, all computed Pareto optimal solutions no matter how many are available. Naturally, the more Pareto optimal solutions are available, the better the applicability study is able to notify the DM about too large variations during the trade-off analysis.

Furthermore, in some cases the DM might be interesting in knowing that if the first active Pareto optimal solution obtained through applicability study is dominated by approximated Pareto optimal objective vector then we can suspect that the trade-off plane is giving an outer approximation for the Pareto surface and approximated Pareto optimal objective vectors are infeasible. Same kind of analogy can be considered when the first active Pareto optimal objective vector dominates the approximated Pareto optimal objective vector (inner approximation of Pareto surface). However, it is clear that in the case of highly nonconvex Pareto surface these interpretations can be very misleading especially if the applicability study is carried out using relatively small number of Pareto optimal solutions which are not enough to capture the overall shape of the Pareto surface.

5   Example

In this section, we demonstrate the usage of the trade-off analysis tool by considering a nonconvex problem having three objective functions. This example problem is constructed to study nonconvex Pareto surface with the trade-off analysis tool and therefore the objectives do not have any practical interpretation. Even thought the example problem does not contain any computationally demanding functions, it must be pointed out that while carrying out the trade-off analysis we do not need to evaluate functions related to the original problem.

In this example, the objective functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2, 3)$ are constructed by using function $\phi(x_1, x_2) = \psi(x_1, x_2) - exp(-50\psi(x_1, x_2))$, where the nested function is $\psi(x_1, x_2) = x_1^2 + x_2^2$. The actual problem is expressed in the form

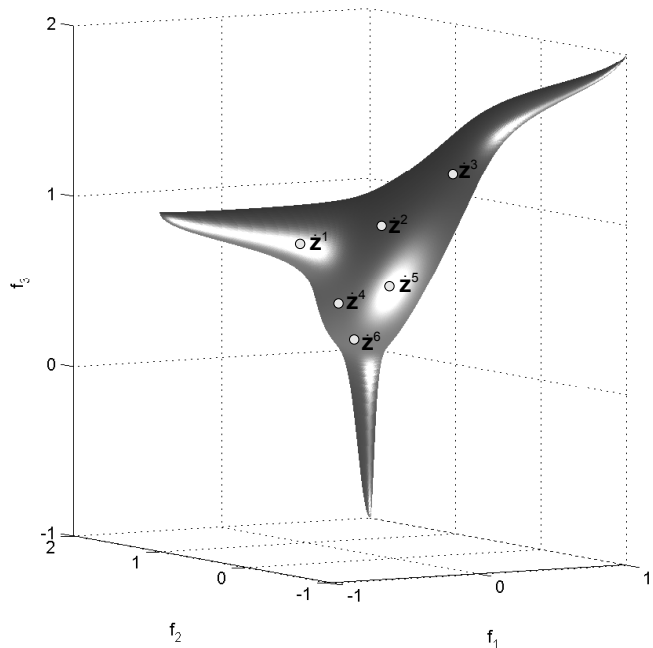|	f_1	f_2	f_3
$\boldsymbol{z}^{\text{low}}$	-1.00	-1.00	-1.00
$\boldsymbol{z}^{\text{up}}$	1.00	2.00	2.00
$\dot{\boldsymbol{z}}^1$	-0.18	0.78	0.78
$\dot{\boldsymbol{z}}^2$	0.16	0.41	0.91
$\dot{\boldsymbol{z}}^3$	0.47	0.12	1.22
$\dot{\boldsymbol{z}}^4$	0.16	0.91	0.41
$\dot{\boldsymbol{z}}^5$	$\mathbf{0.28}$	$\mathbf{0.53}$	$\mathbf{0.53}$
$\dot{\boldsymbol{z}}^6$	0.41	1.16	0.16

Figure 4: Pareto optimal solutions

Figure 5: Objective space

$$\text{minimize} \quad \{f_1 = \phi(x_1, x_2), f_2 = \phi(x_1, x_2 - 1), f_3 = \phi(x_1 - 1, x_2)\}. \qquad (4)$$

Let us assume that the DM has already obtained six Pareto optimal solutions by using an interactive method. The objective function values of these solutions are shown in Figure 4 together with upper and lower bounds for objective functions. In this case the DM has determined bounds $\boldsymbol{z}^{\text{up}}$ and $\boldsymbol{z}^{\text{low}}$ by using his/her expertise. The DM has selected the Pareto optimal objective vector $\dot{\boldsymbol{z}}^5$ for trade-off analysis. The Pareto surface and the six Pareto optimal solutions are visualized in Figure 5. Even thought the Pareto surface is visualized in Figure 5 its explicit form is assumed to be unknown [1].

As already explained in Section 3, the DM is able to vary the components of the Pareto optimal objective vector selected to produce a reference point which is then projected to the trade-off plane. Let us assume that we have a simple user interface containing bar chart visualization for each Pareto optimal objective vector computed and the DM has selected Pareto optimal $\dot{\boldsymbol{z}}^5$ for the trade-off analysis (the upper part of Figure 6). In case (a) appearing in Figure 6, the DM is interested in studying what kind solutions are approximately available if (s)he wants to improve the value of f_1 around 5% and value of f_3 around 10% from $\dot{z}_1^5$ and $\dot{z}_3^5$, respectively (percentage referes to ranges $(z_i^{\text{up}} - z_i^{\text{low}})$, i=1,2,3). To compensate this improvement the DM is willing to sacrifice around 5% in the value of f_2 from $\dot{z}_2^5$. The DM proposes this variation by adjusting the position of gray triangle markers appearing in (a) bar

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[1]Note that this kind of a visualization is not possible with more than three objectives. Furthermore, with practical problems given in a nonanalytic form, it may be computationally very time-consuming to produce visualizations such as in Figure 5.
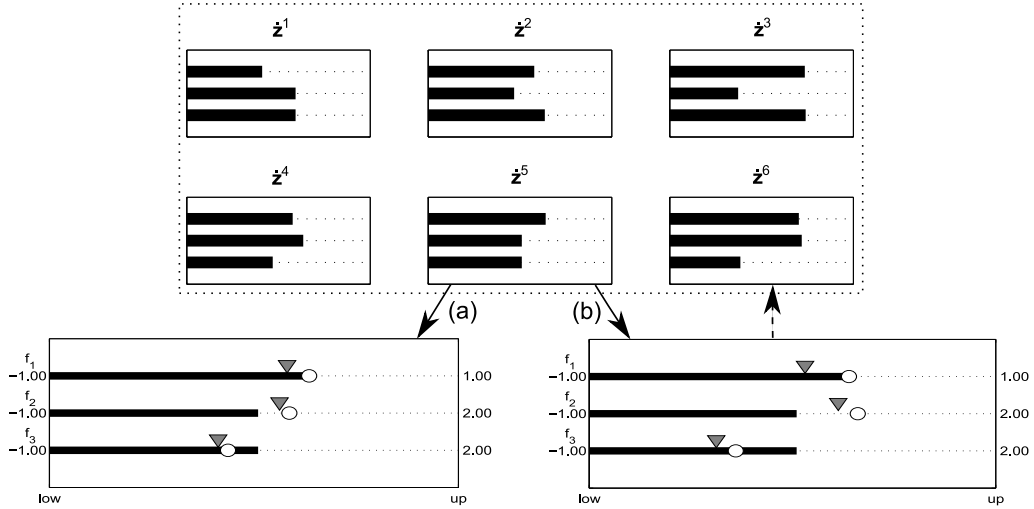
Figure 6: Trade-off analysis with applicability notification

chart in Figure 6. The resulting approximated Pareto optimal objective vector is $\tilde{z}$ and it is indicated in Figure 6 with white circles. By using Algorithm 1 we can confirm that the approximation $\tilde{z}$ can be considered realistic and applicable with respect to the five other Pareto optimal solutions computed. However, we do not know how far from the Pareto surface P the approximated Pareto optimal objective vector $\tilde{z}$ really is.

In Figure 6, case (b) reflects the situation where the DM has proposed a twice as large variation as in case (a). The trade-off between objective functions remains therefore unchanged, but the difference is that now the reference point $\bar{z}$ produces the approximated Pareto optimal objective vector $\tilde{z}$ which dominates the Pareto optimal objective vector $\dot{z}^6$ and, thus, the approximation is not realistic. By using Algorithm 1 we can confirm that $\dot{z}^6$ is the first Pareto optimal objective vector which becomes active when we move from $\dot{z}^5$ towards $\tilde{z}$. Because the approximation obtained is not realistic the applicability study suggests for the DM that maybe (s)he would like to do trade-off analysis based on the Pareto optimal solution $\dot{z}^6$ instead of the current solution $\dot{z}^5$. However, it must be emphasized that this is only a suggestion and the DM is allowed to continue the analysis even thought the resulting approximations are not realistic. The approximated Pareto optimal vectors can be used by the DM to study the local trade-off structure around the selected Pareto optimal solution even thought it might lead in practice to an unrealistic approximation.

The analysis carried out can be used, for instance, to determine explicitly the next reference point which is then utilized in problem (2) to produce the next Pareto optimal solution. On the other hand, the trade-off tool can be also used to determine a promising classification in a classification based method. In such a case the DM is able to grasp and idea what kind of changes to expect when a certain kind of classification is made. This is especially useful when we are dealing with problems having several objectives.

6 Discussion

The definite strength of the method proposed is that the linear approximation of the Pareto surface can be produced as a byproduct when a Pareto optimal solution is computed. Furthermore, in the trade-off analysis, all computations are fast and can be performed in real-time. Because of this, the DM is able to capture, with a relatively small effort, an idea about the trade-off rates and local behavior of the Pareto surface around the selected Pareto optimal solution.

The reference point determined in the trade-off analysis can be used directly to produce the corresponding actual Pareto optimal solution. Therefore, this tool can be used to build up confidence for the DM while (s)he is setting the next reference point in a reference point based interactive method. However, the DM is not forced to set preferences according to the information obtained in the analysis. This method can be seen as a purely additional decision support tool that can be used in an interactive framework when needed. The benefit of this approach is that the trade-off analysis can be made dynamically through visualization, like presented in Figures 2 and 6, instead of considering just static numerical trade-off values.

On the other hand, one potential drawback related to the trade-off analysis tool is the quite demanding assumptions that must be satisfied when solving problem (2), in practice it might be difficult to check that assumptions in Theorem 3.1 hold. For instance, the second order KKT sufficiency conditions assume that the functions related to the problem are twice continuously differentiable. This may be difficult to guarantee in the case of practical problems. Furthermore, linear independence of active constraints may also be problematic if we have, for instance, more objectives than variables (however, this happens very rarely in practical applications). We must also emphasize that the solver used to solve problem (2) must be able to produce KKT multipliers.

As a second potential drawback we can consider the lack of accurate indicator how well the linear approximation obtained by using trade-off plane at some Pareto optimal solution is able to capture the shape of the Pareto surface. In other words, how large variations the DM is able to propose to the Pareto optimal solution analyzed in such a way that the obtained approximation is still in some sense realistic (an approximation error). In the case of highly nonlinear Pareto surface the trade-off information obtained might in the worse case hold only in an insignificantly small neighborhood of the Pareto optimal solution considered. The applicability study presented can help in indication of unrealistic approximations but it works efficiently only when we have enough Pareto optimal solutions already computed. However, it is generally known that typically in interactive methods the DM is carrying out only relatively small number of iterations which might mean that a set of Pareto optimal solutions available for applicability study is quite small. In other words, available Pareto optimal solutions are dominating relatively small set of objective vectors in the objective space. This kind of problem is also related to case where we have large number of objectives.

In addition to the approximation error issue mentioned, one drawback that can be also related to the trade-off plane approximation methods is the problem whether the Pareto surface exists "under" the trade-off plane when we move at the plane to

the certain direction (consider trade-off analysis in a very narrow area of the Pareto surface like for instance in some of the "spike-shaped" areas in Figure 5). In general this situation is difficult to determine in the case of the general nonlinear problem.

Let us point out that the solution method can also produce a representative set of Pareto optimal solutions before actual interaction with the DM. The representative set is assumed to capture the shape of Pareto surface (so called a posteriori methods can be utilized, see e.g. [6]). Then DM is allowed to study trade-offs at Pareto optimal solutions belonging to the representative set generated. Because of more Pareto optimal solutions are available the applicability study proposed can more efficiently point out unrealistic approximated Pareto optimal solutions for the DM. This kind of utilization of the trade-off tool can be, for instance, seen as a part of the Pareto navigator method [4] where the DM is first building an intuition (learning phase) related to available Pareto optimal solutions by using an piecewise linear approximation of Pareto surface based on a generated representative set of Pareto optimal objective vectors. When some interesting area of Pareto surface is located the DM can continue to analyze the selected Pareto optimal solutions more closely (decision phase). Here the trade-off analysis tool can be utilized.

Topics for further research include a study of efficient applicability and error measures that could be conveniently related to the approximation used for trade-off analysis. The aim is that applicability measures developed should have some concrete meaning for the DM and the analysis should be made possible without needing to increase computation workload too much. For instance, it must be studied further how all Pareto optimal solutions produced during an interactive procedure, at the moment when the trade-off analysis takes place, could be used to produce applicability measures. In addition, an approximation error between the trade-off plane and the Pareto surface is also an important topic. Furthermore, the proposed tool must be tested with real problems and decision makers.

7   Conclusions

We have introduced a trade-off analysis tool for supporting a DM in using interactive reference point or classification based methods in nonlinear multiobjective optimization. When using such methods, the DM is supposed to indicate one's preferences of how to improve the Pareto optimal solution under consideration by specifying aspiration levels forming a reference point or a classification for the objective functions. Typically, the DM is not supported in this preference specification task. However, in particular when the problem involves computationally costly functions, it may be time-consuming to wait for a new solution corresponding to the specified desires to be computed. Our tool provides support in this respect. In other words, we provide an approximated solution and the DM can then judge whether it is worthwhile to take the time and generate a new Pareto optimal solution corresponding to the preferences specified or whether it is better to change the reference point or classification. In this way, the DM can avoid unnecessary trial-and-error in obtaining satisfactory solutions.

The approximated solution is obtained using a local, linear approximation of the

Pareto optimal set (this approximation is formed at the point under consideration). The tool projects the reference point specified onto the linear approximation. This can be done with a minimal computational effort. Importantly, the tool utilizes information contained in Karush-Kuhn-Tucker multipliers which can be obtained as byproducts when Pareto optimal solutions are generated. This means that we do not need to produce any additional information in order to carry out the analysis. Thus, the tool is particularly useful for computationally costly problems.

Besides the trade-off analysis tool, we propose a supplementary applicability study which can be used to analyze how realistic the outcome of the trade-off analysis is. The idea is to compare the outcome to other Pareto optimal solutions available (which have been generated earlier during the solution process). In this way, we take advantage of the information available and improve the reliability of the trade-off analysis.

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