Consumer-owned retail cooperative in duopoly with horizontally differentiated goods: a Finnish experience

Economics

Master's thesis

Tuomas Kronqvist

2010

Department of Economics

HELSINGIN KAUPPAKORKEAKOULU

HELSINKI SCHOOL OF ECONOMICS

HELSINKI SCHOOL OF ECONOMICS

Department of Economics

CONSUMER-OWNED RETAIL COOPERATIVE IN DUOPOLY WITH HORIZONTALLY DIFFERENTIATED GOODS: A FINNISH EXPERIENCE

Master's Thesis in Economics

Tuomas Kronqvist

(Spring Term) 2010

Approved by the Head of the Economics Department ___/___ 201___ and awarded the grade ____

HELSINKI SCHOOL OF ECONOMICS
Master's Thesis in Economics
Tuomas Kronqvist

ABSTRACT
9.4.2010

CONSUMER-OWNED RETAIL COOPERATIVE IN DUOPOLY WITH HORIZONTALLY DIFFERENTIATED GOODS: A FINNISH EXPERIENCE

In this thesis I analyze the prevailing market structure and a peculiarity of the Finnish grocery retail trade. The market structure resembles duopoly. The two biggest retailers, S-group and K-group, have market shares of 43 and 34 per cent, respectively. Grocery retail trade is exceptionally concentrated in Finland although other Nordic countries also have a concentrated market. The peculiarity of the Finnish market is that the market leader, S-group, is not a profit-maximizing firm but a consumer-owned cooperative. Instead of shareholders the group has 1.8 million member-owners. I formulate and analyze the Nash equilibrium of a duopoly with a cooperative and a profit-maximizing firm (mixed duopoly). I also compare the resulting equilibrium to the Nash equilibrium of a duopoly with two profit-maximizing firms (normal duopoly).

I perform the analysis by constructing a duopoly model with horizontally differentiated goods using a framework developed by Singh and Vives (1984). I also build on the work of Anderson et al (1979) and Ireland and Law (1983). In the basic model employed in this thesis, firms are assumed to compete in price (Bertrand competition) and all consumers are assumed to be homogenous members of the cooperative. However, after analyzing this basic model I separately consider how equilibrium changes if the cooperative were a Stackelberg leader and if an exogenous part of consumers were not part of the cooperative.

The cooperative chooses to price at marginal cost in the Nash equilibrium of mixed duopoly. This is significant because in normal duopoly firms use market power to sustain above marginal cost pricing. The cooperative's competitor, a profit-maximizing firm, can still sustain above marginal cost pricing in mixed duopoly but less so than would be possible in normal duopoly. In Stackelberg competition the cooperative chooses to price below marginal cost. If some consumers are not members of the cooperative the firms charge different prices due to different reaction curves with the cooperative having the higher price.

The willingness of consumers to substitute one good for the other and their value for variety is measured in the model by parameter gamma. A high level of gamma means that consumers are ready to substitute one good with the other and that they do not value consuming variety. In normal duopoly consumer utility is the higher the higher gamma is. In mixed duopoly utility is highest if gamma is low. The difference is explained by the effect of gamma on pricing power, which is not important in mixed duopoly where the cooperative already restricts the use of pricing power.

If consumers are faced with a choice between normal and mixed duopoly, they are better off choosing mixed duopoly since the cooperative reduces the adverse effects of duopoly. This result has implications on how one should approach the current market structure in Finland since the model's theoretical cooperative has much in common with actual S-group behavior.

Keywords: cooperative, grocery retail, duopoly, Bertrand, differentiated goods

TABLE OF CONTENTS

TABLE OF FIGURES

1. Introduction

”The purpose of a company is to generate profits for the shareholders”

- Limited Liability Companies Act - Finland, section 5 (2009)

”The purpose of a co-operative shall be to promote the economic and business interests of its members”

- Co-operatives Act - Finland, section 2 (2009)

In this thesis I study a duopoly in the grocery trade with horizontally differentiated goods. Instead of two profit-maximizing firms, duopoly players consist of a consumer-owned cooperative and a profit-maximizing firm. As revealed by the above quotes these two firm types have very different objectives. I focus on how the Nash equilibrium of this mixed duopoly compares to a Nash equilibrium of a normal duopoly, i.e. that of two profit-maximizing firms. A consumer-owned cooperative is a firm that is not owned by shareholders but rather by its customers. Each customer-owner has an equal say of how the firm should be organized. Instead of maximizing profit the cooperative’s goal is to maximize individual member surplus. Two major players dominate Finnish grocery trade. S-group and K-group have a combined market share of almost 80 per cent. S-group is a cooperative and K-group is a profit-maximizing firm listed in the Helsinki stock exchange. This market reality makes the question of a how a cooperative changes equilibrium extremely interesting.

I study the mixed duopoly setting by constructing an explicit model and comparing its equilibrium to that of a normal duopoly’s equilibrium. In order to make explicit comparisons I assume a specific consumer utility function. Other simplifications are also made in order to limit the analysis to that of what is appropriate for this thesis. The most important of these is that all consumers are assumed to be homogenous members of the cooperative. In the basic model firms compete by setting price simultaneously, i.e. Bertrand competition. In addition to analyzing this basic Bertrand model I briefly study two extensions. First, I assume that firms compete in price but that the cooperative decides first, i.e. Stackelberg competition. Second, I relieve the restriction of consumer homogeneity and study how equilibrium changes when an exogenous portion of consumers are not members of the cooperative.

The questions I examine in this thesis are all related to the consumer-owned cooperative. This form of firm has been largely disregarded in mainstream economic literature. Questions include: How is a consumer-owned cooperative different from a profit-maximizing firm? Do these differences cause differences in behavior between the two types of firms? What does a duopoly model with these different types of firms suggest as behavior? How does the model relate to the reality of the Finnish market? What are the policy implications?

Duopolies with two profit-maximizing firms have been studied extensively. However, the effect of having a cooperative as a market participant instead of a profit-maximizing firm has not been in the focus of research. Globally, consumer-owned cooperatives in the grocery trade have been rare and perhaps that is why they have attracted limited research interest. However, some research has been done. Ireland and Law (1983) and Anderson et al. (1979) have made important contributions. The former studied a Cournot-Nash model of a consumer-owned cooperative. The latter analyzed the economics of consumer-owned cooperatives in general. Both of the aforementioned papers build on the pioneering work of Enke (1945). This thesis aims to make a contribution to this field.

Throughout the thesis gamma is used to depict consumers' preference for variety. A high level of gamma means that consumers are ready to substitute one good for the other. In contrast, lower levels of gamma imply that consumers prefer variety and are hesitant to substitute. Strictly speaking consumers are born with some appetite for variety that cannot be directly affected by retailers. However, retailers can affect consumer choice by changing the location and variety of their stores. Modeling a retailer's optimal choice problem is not at the centre of this thesis and such considerations will be absent. Having said that, this thesis is interested in how different levels of gamma affect equilibrium prices, quantities and profits. Thus, I adopt a less strict interpretation of gamma in which the parameter is thought to contain both the inherent appetite for variety but also the sensitivity of consumers' willingness (and possibility) to substitute one good for the other. Retailers can indirectly influence the latter by their choice of horizontal differentiation. This interpretation allows me to consider how equilibrium depends on both consumers' taste for variety and, indirectly, retailers' good differentiation choice.

This thesis is composed of nine parts and two appendices. Following this introduction I describe the state of Finnish grocery trade. I take a look at the major players and describe some recent changes in market structure. I find that the market has become more concentrated during the past fifteen years and that S-group has become the market leader ahead of K-group. I then take a look at cooperative theory. I describe in detail what a cooperative is and consider what makes it different from a profit-maximizing firm. I build on Hansmann (1996 and 1999) and discuss the circumstances in which a cooperative might be a better form of institution than a profit-maximizing firm. In the analysis I take the consumer's point of view. I find that if consumers are homogenous enough and the market displays imperfections, e.g. in the case of monopoly, a cooperative can be a good form of institution to organize a firm.

In the fourth part of this thesis I review some aspects of imperfect market theory. Specifically I look at how standard perfect market assumptions on the number of market participants, homogenous goods and profit-maximizing firms differ from Finnish market reality. I use this analysis to shape the model in the fifth part of the thesis.

The fifth part of this thesis is its backbone and contains the modeling of the market setting. I first discuss some limitations and assumptions before building the model. I create the model using the frameworks utilized by Anderson et al. (1979) and Ireland and Law (1983). Additionally, the linear demand system employed by Singh and Vives (1984) is put into use. I formulate the explicit Nash equilibrium of the duopoly setting and also investigate the equilibrium at different levels of consumer's preference for variety (represented as gamma in the models). In order to have a benchmark I present the equilibrium of a duopoly with two symmetric profit-maximizing firms (normal duopoly). The model employed in this thesis is an adaptation of that used by Singh and Vives (1984) to compare price and quantity competition. Formulations of equations are omitted from the main text but can be found in the equations appendix.

In part six I turn to compare the equilibria of mixed and normal duopoly. I find that price is always lower in mixed duopoly due to the cooperative's reluctance to use its pricing power to the disadvantage of its consumer-owners. The price difference is found to be at its greatest at a moderate level of gamma and thus is not monotonous with respect to consumers' preference for variety. However, consumed total quantity is always higher in mixed than normal duopoly. As one might expect, consumer utility is higher in mixed duopoly whereas firm profit is higher in normal duopoly. However, an interesting finding is that consumer surplus is decreasing with respect to gamma. In other words, if one of the duopolists is a cooperative, consumers are increasingly better off the higher their preference for variety is. This is contrary to a normal duopoly where consumers are better off if they are willing to substitute one good for the other, thus decreasing supplier pricing power.

In the seventh part I expand the analysis by first changing the way the duopolists compete and then by allowing cooperative sales to non-members. Changing the competition setting to Stackelberg competition entices the cooperative to price below marginal cost. Exogenously setting some part of the population as cooperative non-members, I study how allowing cooperative sales to non-members affects equilibrium. Making this change gives the profit-maximizing firm a greater possibility to exercise market power.

In the eighth part of this thesis I take a step back and examine how actual S-group behavior fits the model's predictions. I find that S-group's stated goal is compatible with the optimization problem in the model but that because the model is static, important saving and investment decisions are left out of scope distorting the comparison to S-group performance. I discuss at length the way S-group disburses any profit to its members. A difference between modeled and actual behavior is that S-group employs a non-linear schedule instead of a pure pro rata one. In other words, a customer's return from the cooperative is not linear according to her purchases but is increasing. All in all I find the model to be compatible with S-group's behavior. I close this thesis with conclusions in the ninth part. Conclusions are followed by a list of references and an appendix containing equation formulations.

2. Finnish grocery trade

The Finnish Grocery Trade Association (Päivittäistavarakauppa ry, 2009) defines groceries as "in addition to food, consumer goods consumed on a daily basis which are bought while purchasing food. Groceries include food, drink, cosmetics, toiletry, detergents, home paper, cigarettes, magazines and newspapers." The grocery trade is defined as the retailing of these products in a mostly self-service store. It is important to note that large scale grocery stores, i.e. hypermarkets, commonly sell clothes, electronics and sporting goods. These items are not covered in the definition of groceries and are excluded from my thesis.

Throughout this thesis I use market share instead of earnings as a measure of success when appraising firm performance. The first reason is that market share information is readily available and comparable across time and companies. Second, because S-group is a cooperative and K-group a profit-maximizing firm, comparing earnings would be misplaced since S-group is not trying to maximize its profit as K-group is. Third, because S-group and K-group operate across several countries and in various industries I feel reluctant to start breaking down their reporting in order to get a comparable figure for grocery trade in Finland. Thus I will use market share as the defining measure of success in the Finnish grocery trade.

In this part of my thesis I give background information on the Finnish food retail industry. First, I present the major players. Following that I take a look at changes in market dynamics during the past decade or so.

2.1. Major players

In this section I introduce the major players in the Finnish grocery industry starting with the market leader, S-group. This cooperative group is composed of 22 regional cooperatives and one central cooperative (*Suomen osuuskauppojen keskuskunta*, SOK) which provides central corporate functions to the regional cooperatives and coordinates their operations. S-group operates in six fields: grocery stores (57 % of sales[1]), gas stations (14%), department stores

[1] Contains all sales from grocery stores even when sales are not groceries, e.g. clothing sold from hypermarkets

(4%), car showrooms (6%), hotels and restaurants (7%) and agricultural stores (12%). Hired managers run S-group grocery stores. Sales are made to cooperative member and non-members alike. The group has recently entered retail banking offering daily banking services and has some operations in Russia and the Baltics. Although S-group's grocery stores are organized in several different chains I will disregard this detail and refer to the group as a whole. (S-group, 2009b)

S-group is wholly owned by its 1.8 million (S-group, 2009a) cooperative members who are also consumers of the cooperative. Governance of the local cooperatives is arranged so that cooperative members participate in elections every four years to elect members to a body of representatives (*edustajisto* in Finnish). Each member of the cooperative has one vote in the election. The body of representatives elects the administrative council (*hallintoneuvosto* in Finnish) which appoints the CEO of the local cooperative. In addition, the council appoints three to four members to the administrative board which is chaired by the CEO. Governance of the central cooperative, SOK, is administered much in the same way except that the local cooperatives nominate the body of representatives according to their ownership stake in the central cooperative. (S-group, 2007)

Kesko Oyj (hereafter K-group) is a listed company with four divisions: food trade (38% of net sales), building and home improvement trade (30%), home and specialty goods trade (16%) and car and machinery trade (15%). Food trade is present only in Finland but the other divisions have operations in Russia, Sweden, Norway and the Baltic countries. K-group grocery stores are run by retailer entrepreneurs with a personal investment in the store but who are supported and trained by K-group. As with S-group, K-group consists of several different chains but I will treat the whole group as one entity. (Talma, 2009)

In 2009 K-group and S-group held 77 per cent of the grocery market share (see exhibit 1, page 8). The remaining 23 per cent is split among Suomen Lähikauppa Oy with a 10 per cent share, Lidl Ky with five per cent and other smaller players (HOK-Elanto, 2010b). Suomen Lähikauppa Oy, previously know as Tradeka, was formed from the bankruptcy of the EKA-cooperative in 1993 (Seppänen, 1993). Nowadays 66 per cent of Lähikauppa Oy is owned by a Swedish private equity house, IK Investment Partners (Suomen Lähikauppa, 2008). Lidl Ky, which is part of the German hard discounter group, entered the Finnish market in 2002 introducing its Finnish competitors and customers alike to a new way of doing business (Peltola, 2009).

2.2. Developments in market structure

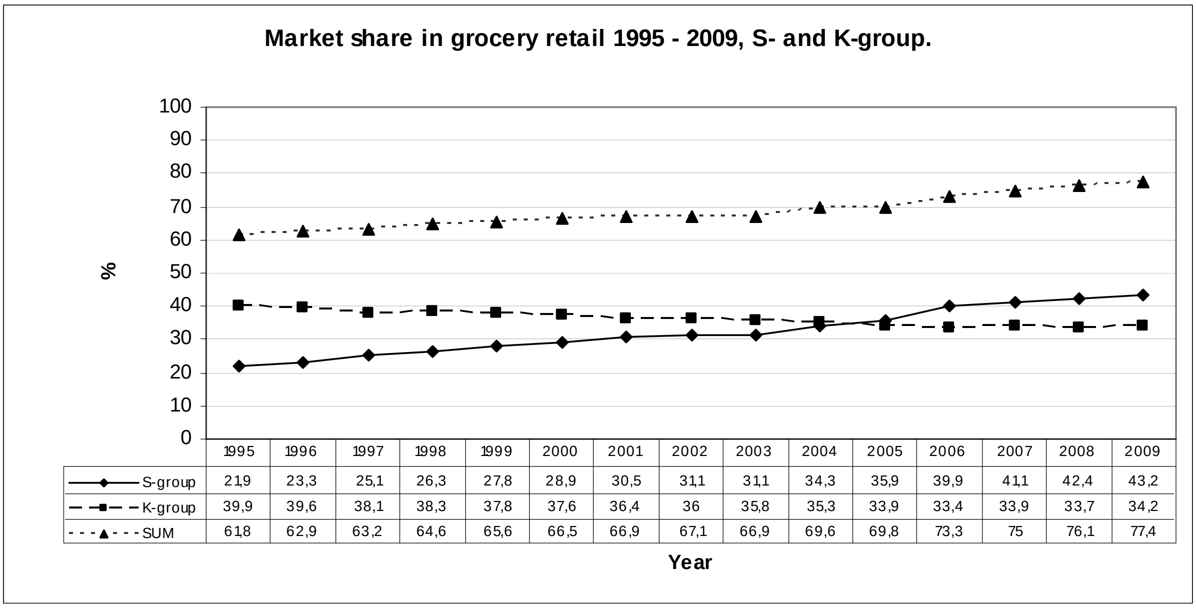
In this section I present the changes that the Finnish consumer goods retail industry has gone through since 2000. I take a quick look at the structure of the industry in 2000, how and why it has changed since then and where it is as the decade ended. Three major points stand out: 1) of the two leading players, S-group has succeeded better and has overtaken the market share of the former leader, K-group, 2) the two groups, K and S, compromise a larger share of the total market than before and 3) alongside these two giants there is a group of smaller firms including Lidl, whose effect on Finnish retailing has been less than anticipated (Peltola, 2009).

In 2000 K-group was the clear market leader (see exhibit 1, page 8) in a shaken up industry. Before the mid-90s the Finnish grocery retail trade was dominated by four large players. Two of them, K-group and Tuko, were profit-maximizing firms whereas the other two, S-group and EKA, were consumer-owned cooperatives. The Finnish depression in the early 1990s changed this landscape and the 21st century began with a setting resembling duopoly with K-group at the lead and S-group 10 percentage points behind in market share.

The next eight years saw S-group stampede past K-group. Peltola (2009) offers several reasons for S-groups prominence and K-group's slide. S-group created a central procurement and logistics company (Inex Partners Oy), focused on four retailer chains and emphasized distribution of retained earnings to cooperative members instead of offering erratic (and often expensive) in-store discounts. At the same time K-group was involved in disputes between its central organization and retailers and was late in restructuring its business to match the quickly urbanized Finnish demography. However, since S-groups market share grew by over 14 per cent and K-group lost only 3.4 in the years from 2000 to 2009, S-group has necessarily gained share also from other retailers. Thus, market concentration has increased since smaller players have also lost market share.

Einarsson (2008) calculated a Herfindal-Hirschman index (HHI) from 2004 data for food retail trade in each of the Nordic countries. Finland's index was 2500. At the time this was the lowest figure for the five Nordic countries in question. Sweden had an HHI of 3100, Denmark 2900, Iceland 2800 and Norway 2600. With new market shares from 2009 Finland's HHI for the four biggest firms is 3200. This increase from 2500 in 2004 to 3200 in 2009 is considerable. To put the Nordic situation in perspective France's and Germany's indices in 2004 were approximately 1600, the UK's 1800 and Spain's between 300 and 500 (Einarsson, 2008). In the US, competition authorities classify an industry with an HHI of 1800 or higher as highly concentrated (United States Department of Justice and the Federal Trade Commission, 1997).

A report by the National Consumer Research Centre (2010, 77) also found the Finnish grocery trade to be highly concentrated. The report cites firm efficiency as one of the reasons for lack of foreign firm entry. Additionally, the report claims that the country's remote location and small market do not entice entry. An exception is of course Lidl. The reports writers highlight that a concentrated industry is seldom in the interest of consumers but do not discuss potential implications further.



	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
S-group	21,9	23,3	25,1	26,3	27,8	28,9	30,5	31,1	31,1	34,3	35,9	39,9	41,1	42,4	43,2
K-group	39,9	39,6	38,1	38,3	37,8	37,6	36,4	36	35,8	35,3	33,9	33,4	33,9	33,7	34,2
SUM	61,8	62,9	63,2	64,6	65,6	66,5	66,9	67,1	66,9	69,6	69,8	73,3	75	76,1	77,4

Exhibit 1 K- and S-group market shares from 2000 to 2009 (Sources: K-group and S-group annual reports; S-group, 2009b; HOK-Elanto, 2010b)

In 2009 the two giants, S- and K-group, had a combined market share of approximately 77 per cent. This is approximately ten percentage points more than in 2000. Thus the Finnish market has become concentrated up to a level which can be described as exceptionally high compared to non-Nordic European countries. A point to note is that concentration has increased throughout the past decade and has not been reversed by the entry of the first real hard discounter, Lidl, in 2002.

What makes the current situation so interesting is that the two major players are so different. K-group is an investor-owned profit-maximizing firm which is listed on the Helsinki stock exchange. The business model of K-group combines a large degree of local retailer independence and entrepreneurship leveraged on a common brand and shared operations. The business model of K-group has much in common with franchising. In contrast, S-group is a consumer-owned cooperative. Local retailers in the group are run by hired managers who have a limited amount of independence in how to run their store. Independence at the regional cooperative level is much higher. However, the relatively high level of coordination is characteristic to S-group and has undoubtedly contributed to its success in the Finnish grocery trade.

These distinctions make an enquiry into the prevailing duopoly setting very interesting. Next I will examine the relevant theory on cooperatives in order to later evaluate if they depict Finnish reality.

3. Cooperative theory

This thesis is an inquiry into cooperative firms in an imperfectly competitive grocery retail industry. In this section I present the relevant literature which I will build my later arguments on. First I define a cooperative and give a brief overview of its history. Then I study the circumstances in which a cooperative might be an efficient way to organize a firm.

3.1. What is a cooperative?

The roots of the cooperative firm can be traced back to the English Equitable Rochdale Pioneers of 1844. These founders of the first cooperative operated according to seven Rochdale principles, which had a major influence on later cooperatives. The influence of their ideals can still be seen today. These seven principles were: (Thompson, 1994)

1. open membership,
2. democratic control (one man, one vote),
3. distribution of surplus in proportion to trade,
4. payment of limited interest on capital,
5. political and religious neutrality,
6. cash trading, and
7. promotion of education.

From Rochdale the cooperative movement spread throughout the world to several different industries. Four industries stand out as areas where the cooperative movement has gained most ground. These are agriculture, banking and finance, insurance and retailing. Examples of these include, respectively: farmers forming a cooperative to market their goods, credit unions, mutual insurance and consumer-owned retailers. Geographically consumer-owned retail cooperatives have been uncommon in the United States but have been popular in some European countries, including Finland. In this thesis I focus on consumer cooperatives in the retail industry. Despite ample research in the fields of farmer producer cooperatives, cooperatives in developing economies and worker cooperatives, there is much less research in consumer cooperatives. (Jones & Kalmi, 2009)

I define a standard consumer-owned cooperative in the retail industry as a firm following seven central principles. Compared to the above Roschdale principles, many characteristics have stayed the same since 1844. If not mentioned separately, I will assume these cooperative characteristics throughout this thesis. A standard consumer-owned cooperative will display the following characteristics:

1. The firm is owned by its members,
2. voting rights are divided equally between members,
3. members are also customers of the firm,
4. disbursed retained earnings are distributed pro rata consumption,
5. the firms maximizes net individual consumer surplus,

6. cooperative can choose the amount of members, and
7. sales to non-members are prohibited.

The first five depict traditional cooperatives and are the key characteristics which differentiate a cooperative from a profit-maximizing firm. The first three are self-explanatory but numbers four and five deserve some thought. Disbursing retained earnings on a pro rata basis is a common feature of cooperatives and is meant to reward those who contribute most to the cooperative. In the case of a consumer-owned retail cooperative, this means that the more a consumer buys from the firm the bigger her share of retained earnings is. Moreover, a given member's share of profit grows in a linear fashion with respect to money spent on the cooperative. The fifth characterization is the most important difference between a profit-maximizing firm and a cooperative. Whereas a profit-maximizing firm only maximizes supplier surplus, the cooperative will maximize the net benefit of each cooperative member. However, it should be noted that a cooperative is not the same thing as a government-owned entity with a goal of maximizing total surplus. Rather, the consumer-owned cooperative seeks to maximize the individual surplus of each cooperative member.

Characterizations six and seven are in order to simplify modeling. They conflict somewhat with Finnish reality. Point six conflicts with the aforementioned Roschdale ideal of free entry. However, allowing the cooperative to choose the amount of members is consistent with Finnish law (Co-operatives Act, 2009). The market leader and only consumer cooperative in the grocery retail trade, S-group, does not restrict entry. One could however interpret that the amount of members in S-group is below or immaterially close to optimum and thus the cooperative chooses not to restrict entry although it has a legal right to do so. There could also be other, non-financial reasons for not limiting entry. In the basic model I will assume that all consumers are part of the cooperative. This is warranted because consumers are assumed to be homogenous. If it makes sense for one to join, all should join. However, I will later expand the basic model to consider a situation where part an exogenously determined part of the population is not part of the cooperative.

Point seven is in stark conflict with the reality of S-group but is in line with United States law which does not allow sales to non-members. From a Finnish point of view, US legislation is unfortunate since American research in the field often takes point seven for granted even though this is not the case in Finland. Being well aware of this contradiction of reality, I choose to first uphold this restriction in basic modeling. I will however later expand the basic model and consider how sales to outside members changes equilibrium.

I have now presented a short history of the cooperative movement and depicted how a firm organized as a consumer-owned cooperative is defined. I turn to an obvious question: why cooperatives exist at all. Since privately owned profit-maximizing firms are generally optimal for society an exception to this rule needs to be motivated.

3.2. When would a consumer-owned cooperative be more efficient?

To answer this question I need explore how a cooperative actually differs from a profit-maximizing firm. Then, after mapping the differences, I look into the range of circumstances when it might make sense for consumers to join forces and form a consumer-owned cooperative instead of relying on market transactions.

So how different is a consumer-owned cooperative retailer from a typical profit-maximizing firm? Conceptually, there are surprisingly few differences. Hansmann (12, 1996) argues that typical profit-maximizing firms are in fact just another type of a cooperative, a lenders' cooperative. The only transaction of a firm's lenders is to supply capital and thus they are only interested in collecting a profit on their investment: lenders' cooperatives optimize return on investment. In contrast, the primary transaction of a firm's customers is buying from the firm. Thus they are interested in the net cost of their shopping. From this observation and the earlier mentioned seven characterizations of a consumer-owned cooperative follows that the most important difference between the two firm types is that they optimize different outcomes.

The key question of consumer cooperatives is whether they are more efficient way of organizing a firm and if yes, in what circumstances. In Hansmann's words (1999): "Efficiency is best served if ownership is assigned so that total transactions costs for all patrons, including both costs of market contracting and costs of ownership, are minimized". By patrons Hansmann means "all of the firm's customers and suppliers" including suppliers of labor and capital. So how is efficiency, measured as firm costs, affected by whether the firm is a cooperative or a profit-maximizing firm? Input costs should be the same for both institution types. It is fair to assume that consumer-owned retail cooperatives pay the same price for their supplies as other types of firms. This leaves other aspects of the cost function. Hansmann (1999) defines a categorization of costs which presents a methodology for evaluating when total transaction costs for all patrons could be lower for a consumer-owned retail cooperative than for an otherwise identical profit-maximizing firm.

The first category includes costs of market contracting. Essentially this means that it could be beneficial for consumers to have direct control over their supplier through ownership rather than relying on simple market transactions, i.e. negotiating a price and buying from the supplier. Monopoly power of the supplier could be an obvious motive for consumer ownership. Another could be the presence of asymmetric information. In Hansmann's (1999) example, a supplier may be better informed of the quality of products it sells and it can be overly costly for the consumer to control the quality of its supply. In such a case it can be beneficial for the customer to have ownership in the cooperative and have power over the product itself.[2]

---

[2] Interestingly enough, these examples of monopoly and asymmetric information point to well known problems of imperfect competition. Thus consumer-owned cooperatives and cooperatives in general could be seen as tools of mechanism design in which a market failure or imperfection can be addressed and potentially resolved.

The second category consists of ownership costs. Hansmann (1999) divides these costs into three groups: costs of monitoring, costs of collective decision-making and costs of risk-bearing. The first of these is related to the principal-agent problem of aligning owner and employer goals. The second compares the cost of taking part in collective decision-making versus entering into a market transaction. The third is the cost of capital cooperative members incur when they take on risk on their cooperative membership fee, which may or may not yield profit in the form of cheaper goods or payout of retained earnings.

The cost of risk-bearing is analogous to costs of capital. This costs stems from paying a membership fee to the cooperative which differs from an investor's investment in a profit-maximizing firm. First, memberships are generally non-tradable and redemption policies can be designed to discourage exit. This locks in the members' capital, which is a polar opposite to investing in a listed profit-maximizing firm's stock, which can be freely traded. Second, membership benefits including price reductions on purchases and disbursed retained earnings carry a risk. A cooperative member cannot be sure of the cooperative's ability to offer price reductions or disburse earnings. Whether this risk is higher than the dividend risk and price volatility of a stock is an interesting question. In any case, the importance of risk-bearing costs is surely increasing in membership fee size.

In S-group, initial membership fees vary according to which regional cooperative the member wishes to join. The regional cooperative in the Helsinki area charges 35 euros as its membership fee. However, most regional cooperatives charge 100 euros for joining (S-group, 2010). These are still somewhat modest sums, especially when taking into account that the member does not typically need to pay the whole sum up front. Thus the costs of risk-bearing for these small invested amounts are unlikely to be material.

The other two costs of ownership, costs of monitoring and collective decision-making, can be significant in consumer cooperatives, especially small ones. Participating in collective decision-making and monitoring the actions of the firm can have considerable costs associated to them. If the goals of cooperatives consumers are homogenous enough, collective decision-making and costs of monitoring can be thought of as a fixed cost. From this follows that cost per member is declining with respect to the number of cooperative members. However, if members have different goals this relation can be largely offset and even reversed as the number of members increases. Heterogeneity increases costs of ownership for all.

Holmström (1997) also emphasizes the importance of homogeneity and possibility of exit in collective decision making. The two are closely interlinked. If there is a mechanism of exit in the firm, it serves to homogenize the owner base. If a shareholder disagrees with the management of a listed profit-maximizing firm she can either use her voice to change the firm's strategy or she can exit by selling the stock. Even the credible threat of exit is important for shareholders to get their voice heard. Managers often listen to shareholders not because they fear active shareholder action but because they can exit the company by selling stock and thus pushing the share price down. In fact, Hansmann (1999) argues that the reason for the prevalence of investor-owned profit-maximizing firms in our modern economy has everything to do with homogeneity. He argues that lenders of capital have the most homogenous interests of maximizing their return on investment. Thus, organizing firms as lenders' cooperatives is commonly the most efficient arrangement.

In addition to these cost of ownership and market contracting, there are also non-financial reasons why a cooperative might be a good way of organizing a firm. An example of these non-financial arguments for cooperatives is valuing consumer participation in the firm's democratic decision-making in itself (Hansmann 1996). However, I choose to disregard these considerations and focus entirely on efficiency aspects. I do this for two reasons. First, I want to limit the scope of this thesis. Second, valuing the non-economic effects is essentially a discussion about values. This thesis is interested primarily in efficiency and understanding the economic dynamics of cooperatives in a duopoly setting. However, excluding such benefits from this thesis does not mean that non-economic considerations are unimportant.

Summarizing, one can say that from a consumer's point of view there are circumstances in which forming a cooperative might be beneficial. The existence of a large enough homogenous consumer base and market imperfections such as monopoly power can be expected to promote forming a consumer-owned cooperative. Likewise, members must have a low required rate of return for their membership fee in order to entice them to lock-up their investment in the cooperative when alternative liquid investment opportunities are abundant. If the membership fee is sufficiently low, costs of risk-bearing are likely to be immaterial

Next I look at how assumptions about perfect markets need to relaxed in order to make a meaningful analysis about a consumer-owned cooperative in the Finnish retail trade.

4. Imperfect market theory

No real world market meets the requirements of a perfect market characterized by the absence of entry and exit barriers, perfect information, costless transactions, an infinite amount of market participants, profit-maximizing firms and homogenous goods. However, these assumptions are often utilized to enable meaningful research in economics

In studying the Finnish food retail industry some perfect market assumptions need to be relaxed while others upheld in order to make a meaningful analysis. The number of market participants is fixed to two and at least some degree of good heterogeneity is assumed. Also, the assumption of all firms maximizing profit needs to be relaxed in the case of the cooperative firm. This is of course central to my thesis. However, perfect market assumptions of perfect information and costless transactions are upheld. Without these simplifications the analysis would become much more complicated without adding value to the analysis at hand. Next I will discuss these central assumptions underpinning later modeling.

4.1. Number of market participants

First I will discuss the finite number of participants in the Finnish food retail market. As discussed earlier, the market is currently characterized by two large firms who are accompanied by smaller competitors. This is contrary to the perfect market assumptions in which agents are assumed to be small price-takers. The market is not characterized by a perfect duopoly either. However, since my focus is on cooperative firm behavior against a profit-maximizing competitor, I choose to ignore the smaller firms in the market and assume a two-firm setting. I also disregard any entry and exit considerations.

The two classical models of duopoly were separately written in the 19^{th} century by Antoine Augustin Cournot and Joseph Louis Francois Bertrand. The key difference between the two models is that in Bertrand competition firms compete by setting prices and in Cournot by setting quantity. Firms are typically barred from cooperation and equilibrium is resolved by utilizing Nash’s non-cooperative equilibrium concept. (Singh & Vives, 1984)

Which of the classical models depicts competitive behavior in the grocery retail trade? Do firms compete by setting price (Bertrand competition) or quantity (Cournot competition)? Naturally reality is somewhere between these two perfect cases. Retailers may pre-commit to certain volumes and determine price accordingly. However, customers of grocery retailers are cost conscious and the blitz of marketing coupons, temporary price reductions and other pricing related promotions are clear signs of competition via pricing decisions. Thus, the nature of the oligopolistic competition in grocery trade would seem to be more Bertrand than Cournot. Firms compete in prices and let quantity vary accordingly.

A common assumption for both Bertrand and Cournot competition is that firms move simultaneously. A third type of game is a Stackelberg game where one of the duopolists moves first and is called the Stackelberg leader. The Stackelberg follower observes the leader's commitment to some action and only thereafter makes her own move. This kind of setting might be appropriate for a market where one participant has achieved first-mover status. Although I will employ Bertrand competition in the basic model I will later in this thesis change the market setting to Stackelberg in order to study how it affects equilibrium.

4.2. Homogenous goods

Second I look into the perfect market assumption of good homogeneity. Goods can be different in two dimensions: vertically and horizontally. In the former, consumers are able to rank goods in quality order from best to worst. However, they differ in their willingness to pay for the different goods. An example of vertical differentiation is the ranking of cars into luxury, premium and compact. All consumers agree that a luxury car ranks before a premium one but they differ in their willingness to pay for one. Goods can also be horizontally differentiated. This means that consumers have different tastes and are not able to rank goods in quality order. An example of horizontal differentiation is arranging cars into different categories by color. Consumers agree that that colors are different but cannot agree on which color is best.

Although there are premium grocery stores around, e.g. Stockmann Oyj in Finland, in this thesis horizontal differentiation will take central stage since this is how K- and S-group mostly differentiate their goods and services. The most important building block of horizontal good differentiation is Hotelling's spatial model. Consumers are thought to be situated on a line segment. Producers choose their location and thereafter compete in price. Consumers incur a cost when buying from a producer which is not situated at the same point as they are. The cost is increasing in length. Thus, to minimize cost, consumers always buy from the producer closest to them. The equilibrium of producers' location is that they locate to the centre of the line segment. This spatial model can be also expanded to explain other differentiating factors than spatial. For example, the locations on the line segment could be interpreted as differences in good color and a consumer's location on the line segment as a preference for the respective color. (Hotelling, 1929)

There are some obvious ways for a grocery retailer to horizontally differentiate his good. The first and most important one is spatial, i.e. the retailer can decide where to locate her store. A study by The Finnish Grocery Trade Association's (Päivittäistavarakauppa ry, 2009) found that store proximity is the most important factor in a consumer's choice of where to shop. Another way to differentiate is by in-store service. Some consumers might prefer the type of service in one store to that of another store. Other ways of differentiating include branding the store in a way to retain customer loyalty and creating a loyalty program with the purpose of influencing the customer's choice of retailer.

4.3. Profit-maximizing firms

Firms in economic modeling are typically assumed to be profit-maximizing. This is a reasonable assumption since firms strive to maximize shareholder value via dividends or stock repurchases. Although many firms state additional goals such as protecting the environment and being a responsible employer, profit maximization is still their main goal as depicted in the Limited Liability Act (2009): ”The purpose of a company is to generate profits for the shareholders”. The firm's optimization function is thus its revenue reduced by its costs. In a perfect market a firm faces a market price which is unaffected by its own actions. The firm's choice is limited to choosing the quantity it sells on the market.

Since this thesis is interested in cooperatives, it is obvious that the above assumption needs to be adjusted. In a way, cooperatives have the same goal as profit-maximizing firms: they seek to optimize their owners' utility from the firm. However, because owners of the consumer-owned cooperative have an additional role as customers, this has to flow through to the formal optimization formula. Cooperatives have two ways of disbursing money to their customer-owners: return on the membership fee and reduced price. Without loss of generality I will assume that a cooperative can forecast its costs and demand accurately and doesn’t retain any earnings. It sets its price so low that it doesn’t make any profit.

5. Formal framework and modeling

In this section I will build on this thesis's previous discussion to construct a formal duopoly framework consisting of a profit-maximizing firm and a consumer-owned cooperative. I will use the framework built in this section to find an explicit Nash equilibrium. First I discuss some limitations and background of the model.

5.1. Limitations and background

There is a limited amount of differences between a profit-maximizing firm and a consumer-owned cooperative. Indeed, Hansmann (12, 1996) argues that the profit-maximizing firm is actually just another form of a cooperative, a lenders' cooperative. As discussed earlier, one reason for this form's prevalence is due to the high degree of homogeneity among lenders. This homogeneity makes them the patrons whose costs of ownerships are commonly the lowest.

From a modeling point of view Hansmann's insight is important because it gives backing for modeling a cooperative with only a few differences to a typical profit-maximizing firm. For example the cost function can be assumed to be the same since I can expect the differences in logistics, procurement, management and other processes to be immaterial between different forms of organizing the firm. The most important distinction in modeling cooperative behavior compared to that of a profit-maximizing firm is that the two optimize different goals. Whereas the sole goal of the profit-maximizing firm is to maximize profit, the consumer-owned cooperative optimizes the individual net surplus of its members.

Enke (1945), Anderson et al. (1979) and Ireland and Law (1983) have made important contributions to cooperative modeling. Enke looked at the welfare implications of the cooperative and Anderson et al. built on his analysis and expanded it. Ireland and Law constructed a model involving a consumer-owned cooperative in a Cournot-Nash setting. Next I use their research to present a model of a cooperative firm in a duopoly with a profit-maximizing firm.

The basic model in this thesis has three important characterizations: limitation to one period, no sales to non-members and modeling the cooperative as a perfect proxy for its member desired action. As with the previous research, the model displayed here will be a single-period one. Cooperatives and investment behavior is not at the focus of this thesis and thus multi-period considerations are omitted. However, this is not to say that this line of research would be uninteresting. On the contrary, the rapid expansion of the Finnish market leader S-group could be an interesting topic to investigate from a saving and investment behavior point of view.

Second, a relevant question in cooperative modeling is whether the cooperative is allowed to sell to non-members. As discussed before, legislation in the United States has prohibited sales to non-members which is probably the reason why most researchers have taken this for granted. Anderson et al. (1979) developed their basic model assuming sales to members only but do briefly consider how sales to non-members might affect equilibrium. They find that sales to non-members increase cooperative member utility. However, this result cannot be directly applied to the setting of duopoly since Anderson et al. assume that the cooperative's pricing decisions has no effect on its competitors. I will first formulate a model which does not incorporate sales to non-members. Later in the thesis I will extend it to consider sales to non-members by exogenously setting some part of the population as non-members.

Third, if members of the cooperative are assumed homogenous enough, we can model the actions of the cooperative as actions of a representative single member. The underlying assumption here is that the democratic process produces a strategy for the firm which aligns the goals of each homogenous member and the cooperative as a whole. Anderson et al. (1979) use a consumer-manager concept to illustrate how homogenous members choose a member to manage the firm to pursue their shared goals. The writers also assume no management costs. Interestingly, this line of thought implies minimal principal-agent problems. However, it is debatable whether large consumer-owned retail cooperatives are really run by an emergent member of the cooperative. Rather, one could assume that professional management is recruited and compensated just like that of a profit-maximizing firm's entailing the usual principal-agent problems between owners and managers. I disregard this problem and assume that cooperative members are homogenous and that they are able to align the cooperative's interests with their own.

5.2. Building blocks

I will now formally define the building blocks of cooperative modeling adopted from Anderson et al. (1979) and Ireland and Law (1983). The population consists of *N* homogenous consumers. Because consumers are homogenous either all or none choose to be part of the cooperative. My starting point here will be to assume that all choose to be part of the cooperative because without the cooperative they would be forced to buy from a monopoly firm making them inherently worse off. For simplicity I will utilize a linear demand system. The existence of differentiated goods is taken as granted. Singh's and Vives's model (1984) of duopoly is used as a starting point and the consumers' utility function is defined as follows.

$$
\begin{aligned}
&U(y,x) = \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 \\
&\textit{where} \quad (i = 1,2, i \neq j) \\
&\alpha_i > 0 \\
&\beta_i > 0 \\
&\beta_i \alpha_j - \alpha_i \gamma > 0 \\
&\beta_1 \beta_2 - \gamma^2 > 0
\end{aligned}
\qquad (1)
$$

Gamma, γ, depicts whether the two goods are substitutes ($\gamma > 0$), complements ($\gamma < 0$), or independent of each other ($\gamma = 0$). The goods in the model are thought to be the same basket of products which are horizontally differentiated, e.g. spatially. Thus it is infeasible that the goods would be something else than substitutes. What follows is that gamma must be positive. Also, gamma cannot be greater than either beta because otherwise the substitute would satisfy demand "better" than the good itself.

An interesting question is whether the cooperative should maximize the sum of the cooperative's utility or that of each member's. I opt for the latter approach. Despite the benevolent nature of cooperatives, it seems implausible that current members would accept new members if the marginal utility to them of doing so was negative even though taking in new members might increase in the cooperative's total utility. The papers by Anderson et al. (1979) and Ireland and Law (1983) adopt the same approach of optimizing individual utility instead of the sum. Additionally, S-group's stated business idea is to "produce services and benefits to cooperative members" (S-group, 2009b) which would lend further weight to optimizing current individual member utility instead of the total utility of the cooperative.

The cost function of both firms is assumed to display constant marginal costs and no fixed costs. The latter assumption is justified because my focus is not on entry and thus fixed costs play little role in such a setting. Constant variable costs are desirable for a well behaved model. Moreover, there is also some evidence showing that firms in the grocery retail trade do not display increasing marginal costs. Aalto-Setälä (2001) reports that based on Finnish data: "costs per sold unit of large retail entities are not lower than those of smaller firms". It is important to note that Aalto-Setälä is referring to costs at firm level not store level. Thus his finding does not mean that the average cost per sold unit of small stores is the same as in large stores. Because this thesis looks at two firms in a national duopoly, store-specific efficiency factors are secondary to firm-specific factors. Formally I denote marginal cost as constant $c_i : i = x, y$.

$$\begin{aligned} C_x(x) &= xc_x \\ C_y(y) &= yc_y \end{aligned} \quad (2)$$

The price of the good sold by the profit-maximizing firm y, is p_y and the price of the cooperative's good x is p_x. The cooperative's revenue is the product of price, p_x, number of cooperative members, N and the quantity of goods bought by each homogenous member, x. Total cooperative profit, $\prod_c$, is revenues minus total costs. Total profit per member is distributed pro rata according to share of total revenue generated for the cooperative and is depicted as π_c.

$$\begin{aligned} &\prod\nolimits_c = N(xp_x - C_x) // C_x(x) = xc_x \\ &\Leftrightarrow \prod\nolimits_c = Nx(p_x - c_x) \\ &\pi_c = \frac{p_x x}{N p_x x} \prod\nolimits_c = \frac{Nx(p_x - c_x)}{N} = x(p_x - c) \end{aligned} \quad (3)$$

A natural and relevant question is how goods x and y relate to each other. Namely, are they homogenous goods or does some degree of vertical or horizontal differentiation exist? If we consider the reality of the retail industry, it is unlikely that the products on shelf themselves differ to a material degree. A bottle of shampoo from a specific producer has the same features regardless of where it is sold. However, goods are differentiated in many more ways than just product attributes. The most obvious differentiating factor is spatial. Others differentiating factors include retailer brands and different types of service.

Goods *x* and *y* depict baskets of the same products. These baskets do not differ in their product attributes but are differentiated according to the retailer they are bought from, e.g. by dimensions of location. By differentiating their goods retailers can obtain pricing power. The degree to which the two goods are substitutable is a defining factor of market power for both the consumer-owned cooperative and the profit-maximizing firm. To analyze competition with differentiated goods I will apply a linear demand system similar to that of Singh and Vives (1984). Consumer demand for the cooperative's good *x* depends not only on its own price p_x but also on the price of the good sold by the profit-maximizing firm p_y and vice versa.

I now formulate direct demand functions in order to depict the effect of prices on demand. To do this I consider the consumer's standard optimization problem and take partial derivatives of net utility (utility functions minus prices) with respect to the two goods *x* and *y*. This generates inverse demand schedules from which direct demands can be solved. I will later use the demand schedules to find the duopoly setting's Nash equilibrium.

$$
\begin{aligned}
y &= \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \overbrace{\frac{\beta_1}{\beta_1\beta_2 - \gamma^2}}^{b_2} p_y + \overbrace{\frac{\gamma}{\beta_1\beta_2 - \gamma^2}}^{d} p_x \\
x &= \frac{\beta_2\alpha_1 - \alpha_2\gamma}{\beta_1\beta_2 - \gamma^2} - \overbrace{\frac{\beta_2}{\beta_1\beta_2 - \gamma^2}}^{b_1} p_x + \overbrace{\frac{\gamma}{\beta_1\beta_2 - \gamma^2}}^{d} p_y
\end{aligned}
\qquad (4)
$$

Contrary to Singh and Vives (1984), I will not replace the original parameters (alphas, betas and gammas) of indirect demands with price elasticities (denoted b_1, b_2 and d) for the remaining formulation. This is because the cooperative will maximize both consumer utility and profit per member and thus I will need to work with parameters in the consumer's utility function.

5.3. Nash equilibrium

In this part I formulate the Nash equilibrium using the building blocks of the preceding section. The cooperative maximizes the utility of its members' individual consumption. The maximization function contains net utility from consumption and the share of the cooperative's profit. Thus the cooperative's optimization problem is as follows.

$$\max_{p_x} \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_y y - c_x x \qquad (5)$$

Below I present the optimization problem of the profit-maximizing firm.

$$\max_{p_y} \pi(p_y, y) = yp_y - c_y y \quad (6)$$

As argued before, the two duopolists compete by prices in Bertrand competition. Nash equilibrium will be found at the point where the reaction curves intersect. The profit-maximizing firm chooses the price of its good to maximize profit by taking the cooperative's pricing, p_x, as given. It encounters the demand functions for y displayed earlier. The reaction curve of the profit-maximizing firm, RC_y, is as follows.

$$RC_y = p_y(p_x) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \overbrace{\frac{\gamma}{2\beta_1}}^{slope} p_x \qquad (7)$$

The cooperative's reaction curve, RC_x, is more difficult to derive. This is because the cooperative is a special type of firm. Instead of maximizing profit it maximizes the net of consumer utility and share of cooperative profit. The resulting reaction curve is quite interesting. It implies that the cooperative always produces at marginal cost under Bertrand competition.

$$RC_x = p_x = c_x \qquad (8)$$

The sign of γ determines whether the reaction curve of the profit-maximizing good is upward or downward sloping and whether the two goods are strategic complements or substitutes, respectively. Because I have earlier defined the two goods to be substitutes, γ is positive. Thus the goods of the profit-maximizing goods are strategic complements with the restriction that the reaction curve of the cooperative does not depend on the price of the profit-maximizing good's price. To check that the goods are strategic complements I take the second derivative of the profit-maximizing firm's profit function (Bulow et al., 1985).

$$\begin{aligned} \frac{\partial \pi}{\partial p_x} &= \frac{\gamma}{\beta_1\beta_2 - \gamma^2}(p_y - c_y) \\ \frac{\partial^2 \pi}{\partial p_x \partial p_y} &= \frac{\gamma}{\beta_1\beta_2 - \gamma^2} > 0 // \beta_1\beta_2 - \gamma^2 > 0, \gamma > 0 \end{aligned} \qquad (9)$$

Nash equilibrium (p_x^N, p_y^N) is at the intersection of the marginal cost of the cooperative firm (its reaction curve) and the reaction curve of the profit-maximizing firm. Plugging the cooperative's reaction function into the profit-maximizing firm's reaction function I get the Nash equilibrium prices for both cooperative and profit-maximizing firm.

$$\begin{aligned} p_y^N &= \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1}c_x \\ p_x^N &= c_x \end{aligned} \qquad (10)$$

The Nash equilibrium quantities can be solved by plugging equilibrium prices into the direct demand functions.

$$\begin{aligned} y(p_x^N, p_y^N) &= \frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\beta_1\beta_2 - \gamma^2} - \frac{\frac{1}{2}\beta_1}{\beta_1\beta_2 - \gamma^2}c_y + \frac{\frac{1}{2}\gamma}{\beta_1\beta_2 - \gamma^2}c_x \\ x(p_x^N, p_y^N) &= \frac{\beta_2\alpha_1 - \frac{1}{2}\alpha_2\gamma - \frac{1}{2\beta_1}\alpha_1\gamma^2}{\beta_1\beta_2 - \gamma^2} - \frac{(\beta_2 - \frac{\gamma^2}{2\beta_1})}{\beta_1\beta_2 - \gamma^2}c_x + \frac{\frac{\gamma}{2}}{\beta_1\beta_2 - \gamma^2}c_y \end{aligned} \qquad (11)$$

Under these prices the profit of the profit-maximizing firm is as follows.

$$\pi(p_y, y) = \frac{(\beta_1\alpha_2 - \alpha_1\gamma - \beta_1 c_y + \gamma c_x)^2}{4\beta_1\delta} \quad (12)$$

The utility of the cooperative member is as follows ($\delta = \beta_1\beta_2 - \gamma^2$).

$$\begin{aligned} U = {} & \frac{(\alpha_1 - c_x)}{\delta}\left[(\beta_2\alpha_1 - \tfrac{1}{2}\alpha_2\gamma - \tfrac{1}{2\beta_1}\alpha_1\gamma^2) - c_x(\beta_2 - \tfrac{\gamma^2}{2\beta_1}) + c_y(\tfrac{\gamma}{2})\right] \\ & + \frac{\left(\frac{\alpha_2\beta_1 + \alpha_1\gamma}{2\beta_1} - \frac{c_y}{2} - \frac{\gamma}{2\beta_1}c_x\right)}{\delta}\left[\tfrac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma) - \tfrac{1}{2}\beta_1 c_y + \tfrac{1}{2}\gamma c_x\right] \\ & - \frac{1}{2\delta^2}\left[\begin{array}{l} \beta_1\left((\beta_2\alpha_1 - \tfrac{1}{2}\alpha_2\gamma - \tfrac{1}{2\beta_1}\alpha_1\gamma^2) - c_x(\beta_2 - \tfrac{\gamma^2}{2\beta_1}) + c_y(\tfrac{\gamma}{2})\right)^2 \\ + 2\gamma\left((\beta_2\alpha_1 - \tfrac{1}{2}\alpha_2\gamma - \tfrac{1}{2\beta_1}\alpha_1\gamma^2) - c_x(\beta_2 - \tfrac{\gamma^2}{2\beta_1}) + c_y(\tfrac{\gamma}{2})\right)\left(\tfrac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma) - \tfrac{1}{2}\beta c_y + \tfrac{1}{2}\gamma c_x\right) \\ \beta_2\left(\tfrac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma) - \tfrac{1}{2}\beta c_y + \tfrac{1}{2}\gamma c_x\right)^2 \end{array}\right] \end{aligned} \quad (13)$$

5.4. Nash equilibrium at different levels of gamma

In this section I will use comparative statics to see how elements of Nash equilibrium react to different levels of gamma. I focus on gamma because consumers' desire for variety and their willingness to substitute one good for the other are defining factors of retailer competition. As discussed earlier in this thesis, gamma essentially measures consumers' taste for variety, which cannot in principle be changed but is rather a characteristic that consumers are born with. However, a less strict interpretation views gamma as a proxy measure for the consumers' willingness to change consumption from one good to the other. Retailers can affect willingness to substitute by, for example, changing the location of their store[3]. Thus, analyzing Nash equilibrium at different levels of gamma reveals interesting findings about equilibria at different levels of taste for variety and willingness to substitute.

To simplify this analysis, I set $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $c_x = c_y = 0$. The first two assumptions set the demands for the two goods to be symmetric. The intent of this analysis is not to forecast exact demands but rather to map how changes in market structure affect equilibrium. Without this simplification, the analysis would become much more cumbersome

---

[3] However, optimal retailer choice of good heterogeneity will not be covered in this thesis.

without adding much value to the analysis. Setting both marginal costs to zero implies that my focus will not be on potential cost differences between firms and that prices should be interpreted as mark-ups. This has the downside of losing the ability to follow how different cost structures might influence equilibrium. However, the important upside is simplification.

First I analyze prices by taking the derivative of Nash equilibrium prices with respect to gamma.

$$\frac{\partial p_y^N}{\partial \gamma} = -\frac{\alpha}{2\beta} \qquad \frac{\partial p_x^N}{\partial \gamma} = 0 \tag{14}$$

The equilibrium price of the profit-maximizing firm's good is lower the higher gamma is. This is natural since at higher levels of gamma the profit-maximizing firm's pricing power is lower. The price of the cooperative's good is unaffected by changes in gamma because it is always sold at marginal cost.

Next I look at quantities.

$$\frac{\partial x(p_x^N, p_y^N)}{\partial \gamma} = \frac{\partial y(p_x^N, p_y^N)}{\partial \gamma} = -\frac{\alpha}{2(\beta+\gamma)^2} \tag{15}$$

The interesting aspect of quantities is that consumption is lower at higher levels of gamma. This is contrary to how demand would behave in a normal duopoly setting with two profit-maximizing firms, which I will present later. The reason why demand decreases is due to a weakened price effect and a desire for good diversity. In the cooperative setting the price effect of higher gamma is subdued because the cooperative always prices at marginal cost. In a normal duopoly setting this effect would be the main driver of increasing demand at higher levels of gamma. Consumers' preference for diversity is built into the convex indifference surfaces of the concave utility function (Dixit & Stiglitz, 1977). In the mixed duopoly setting with the chosen utility function, the weakened price effect is overshadowed by the loss of utility from variety. Because the consumer gains less from consuming a variety of goods, the consumer demands less of them even though the profit-maximizing firm's good is cheaper.

Now I'll present the effect gamma has on profit.

$$\frac{\partial \pi(p_y, y)}{\partial \gamma} = -\frac{\alpha^2}{2(\beta+\gamma)^2} \quad (16)$$

A higher level of gamma entails lower profit-maximizing firm profit, which is a consequence of decreasing demand and weaker pricing power entailed by a decreased taste for variety.

I also look at how utility changes with respect to gamma.

$$\frac{\partial U}{\partial \gamma} = -\frac{\alpha^2}{4(\beta+\gamma)^2} \qquad (17)$$

As one could expect from the discussion relating to quantity, also cooperative members' utility is declining with respect to gamma. The earlier discussion on quantities applies here as well. The utility-enhancing effect of price on utility is overshadowed by the negative effect of decreased utility from variety. If I was to compare two duopoly settings with different levels of consumers' taste for variety, consumers with a higher taste for variety would reach a higher level of utility.

Last, I look at how total surplus changes with respect to gamma. I have not earlier formulated an equation for total surplus but can derive how it behaves by using the derivatives of profit and utility. Total surplus is the sum of utility and profit and thus its derivative is the sum of its components' derivatives.

$$TS = U + \pi$$
$$\frac{\partial TS}{\partial \gamma} = \frac{\partial U}{\partial \gamma} + \frac{\partial \pi}{\partial \gamma} = -\frac{\alpha^2}{4(\beta+\gamma)^2} - \frac{\alpha^2}{2(\beta+\gamma)^2} = -\frac{3\alpha^2}{4(\beta+\gamma)^2} \qquad (18)$$

As would be expected from its negative components, also total surplus is decreasing with respect to gamma. The interests of market participants are aligned. All would prefer lower values of gamma.

5.5. *The corresponding normal duopoly*

I next formulate the Nash equilibrium of two symmetric profit-maximizing firms in order to have a benchmark for later analysis. To avoid confusion, I will denote the prices, profits and quantities of these two firms with an apostrophe. For example Nash equilibrium price of good x in normal duopoly is denoted by $p_{x'}^{N}$. I will first find Nash equilibrium prices and thereafter profits and quantities.

Prices

$$\begin{aligned} p_{y'}^{N} &= \frac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x \\ p_{x'}^{N} &= \frac{2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_x + \frac{\beta_1\gamma}{4\beta_1\beta_2 - \gamma^2} c_y \end{aligned} \tag{19}$$

Quantity

$$\begin{aligned} y'(p_{x'}^{N}, p_{y'}^{N}) &= \frac{\beta_1}{(\beta_1\beta_2 - \gamma^2)} \frac{2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y}{(4\beta_1\beta_2 - \gamma^2)} \\ x'(p_{x'}^{N}, p_{y'}^{N}) &= \frac{\beta_2}{(\beta_1\beta_2 - \gamma^2)} \frac{2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x}{(4\beta_1\beta_2 - \gamma^2)} \end{aligned} \tag{20}$$

Profit

$$\pi_{y'}(p_{y'}^{N}, y') =$$

$$= \frac{\beta_1}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \left(\begin{array}{l} (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)^2 \\ + (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)(\beta_2\gamma)c_x \\ - \beta_2\gamma c_x{}^2(2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2) \\ - \beta_2\gamma c_x{}^2 c_y(2\beta_1\beta_2 - \gamma^2) - c_x^3\beta_2^2\gamma^2 \\ - c_y^2(2\beta_1\beta_2 - \gamma^2)^2 - c_x c_y \beta_2\gamma(2\beta_1\beta_2 - \gamma^2) \end{array} \right) \qquad (21)$$

$$\pi_{x'}(p_{x'}^{N}, x') =$$

$$\frac{\beta_2}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \left(\begin{array}{l} (2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2)^2 \\ + (2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2)(\beta_1\gamma)c_x \\ - \beta_1\gamma c_x{}^2(2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2) \\ - \beta_1\gamma c_y{}^2 c_x(2\beta_1\beta_2 - \gamma^2) - c_y^3\beta_1^2\gamma^2 \\ - c_x^2(2\beta_1\beta_2 - \gamma^2)^2 - c_x c_y \beta_1\gamma(2\beta_1\beta_2 - \gamma^2) \end{array} \right)$$

Consumer utility under the equilibrium prices is as follows.

$$U =$$

$$= \frac{\beta_1\beta_2}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \left[\begin{array}{l} (2\alpha_1\beta_2 + \alpha_2\gamma - 2\beta_2 c_x - \gamma c_y) \\ * (2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x) \end{array} \right]$$

$$+ \frac{\beta_1\beta_2}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \left[\begin{array}{l} (2\alpha_2\beta_1 + \alpha_1\gamma - 2\beta_1 c_y - \gamma c_x) \\ * (2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y) \end{array} \right] \qquad (22)$$

$$- \frac{\beta_1\beta_2}{2(\beta_1\beta_2 - \gamma^2)^2(4\beta_1\beta_2 - \gamma^2)^2} \left[\begin{array}{l} \beta_2(2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x)^2 \\ + 2\gamma(2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x) \\ * (2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y) \\ + \beta_1(2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y)^2 \end{array} \right]$$

Next I list how the Nash equilibrium of normal duopoly behaves with respect to gamma. Since these are well documented, I do not comment on these further. To simplify, I set $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $c_x = c_y = 0$.

$$\frac{\partial p_{y'}^{N}}{\partial \gamma} = \frac{\partial p_{x'}^{N}}{\partial \gamma} = -\frac{\alpha\beta}{(2\beta - \gamma)^2} < 0 \qquad (23)$$

$$\frac{\partial(y'(p_{x'}^N, p_{y'}^N))}{\partial\gamma} = \frac{\partial(x'(p_{x'}^N, p_{y'}^N))}{\partial\gamma} = \alpha\beta\,\frac{(2\gamma-\beta)}{(\beta+\gamma)^2(2\beta-\gamma)^2} > 0 \qquad (24)$$

$$\frac{\partial(\pi_{y'}(p_{y'}^N, y'))}{\partial\gamma} = \frac{\partial(\pi_{x'}(p_{x'}^N, x'))}{\partial\gamma} = -2\alpha^2\beta\,\frac{\overbrace{\beta^2-\beta\gamma}^{>0}+\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3} < 0 \qquad (25)$$

$$\frac{\partial U}{\partial\gamma} = \alpha^2\beta^2\,\frac{3\gamma}{(\beta+\gamma)^2(2\beta-\gamma)^3} \qquad (26)$$

$$TS = U + 2\pi$$

$$\frac{\partial TS}{\partial\gamma} = \frac{\partial U}{\partial\gamma} + 2\frac{\partial\pi}{\partial\gamma} = \alpha^2\beta\,\frac{-4\beta^2+7\beta\gamma-4\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3} < 0 \qquad (27)$$

Note that consumer surplus is increasing with respect to gamma but that total surplus and profit are decreasing. The reason for a lower total surplus with higher gamma is that the decrease in profit due to a reduction in pricing power is greater than consumers' net utility gain from loss of variety and decrease of price.

6. Different equilibria of the normal and mixed duopoly

I will now compare the prices, quantities, profits and utilities of the two duopoly settings. The goal of this analysis is to find how the two market settings differ and how the level of gamma influences this difference. As before, I set $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $c_x = c_y = 0$ to simplify the analysis. Variables of the normal duopoly are denoted by an apostrophe.

6.1. Price and quantity

By subtracting Nash equilibrium prices from each other I can investigate which one of the two equilibrium prices is higher. For the cooperative's good the answer is clear since the cooperative produces at marginal cost.

$$p_{x'}^N - p_x^N = p_{x'}^N - 0 > 0 \qquad (28)$$

The price difference is dependant on gamma as follows:

$$\frac{\partial(p_{x'}^N - p_x^N)}{\partial\gamma} = \frac{\partial p_{x'}^N}{\partial\gamma} = -\frac{\alpha\beta}{(2\beta-\gamma)^2} \qquad (29)$$

The price difference is decreasing with respect to gamma. Since the cooperative's price is always at marginal cost, this reflects the decrease in the profit-maximizing firm's pricing power due to increasing substitutability between goods. At high values of gamma consumers don't care for variety and are ready to substitute one good for the other thus decreasing the profit-maximizing firm's opportunity to charge a premium for its goods.

For the profit-maximizing firm's good:

$$p_{y'}^N - p_y^N = \frac{\alpha\gamma}{\beta(4\beta^2-\gamma^2)}\left[(\beta-\gamma)(\beta+\frac{1}{2}\gamma)\right] \qquad (30)$$

Whether the difference in prices is negative or positive is defined solely by the equation in brackets since in my assumptions is $\beta^2 - \gamma^2 > 0$, which implies $4\beta^2 - \gamma^2 > 0$. Because $\beta - \gamma > 0$, the price of the profit-maximizing firm's good is always higher in normal duopoly than in mixed. This follows from the way the profit-maximizing firm's room for price increases is higher in normal duopoly where both firms maximize their profit and prices are strategic complements. In contrast, in mixed duopoly the cooperative commits to marginal cost pricing thus limiting the profit-maximizing firm's price.

Next I look at how the price difference responds to different values of gamma.

$$\frac{\partial(p_{y'}^N - p_y^N)}{\partial\gamma} = \overbrace{\frac{\alpha}{\beta(2\beta-\gamma)^2}}^{>0}(\beta - \underbrace{\frac{\sqrt{2}+1}{\sqrt{2}}}_{\approx 1.707}\gamma)\overbrace{(\beta - \underbrace{\frac{\sqrt{2}-1}{\sqrt{2}}}_{\approx 0.293}\gamma)}^{>0} \tag{31}$$

The derivative with respect to gamma shows that the price difference of the profit-maximizing firm's good is increasing when $(0.59\beta > \gamma)$[4] and decreasing when $(0.59\beta < \gamma)$ and has a local maximum at $(0.59\beta = \gamma)$. [5] To understand why, one must recall that I am presenting the price difference between the profit-maximizing firm's good in mixed duopoly and the price of the same good in normal duopoly. The existence of a local maximum is explained by the varying effect of the cooperative's ability to influence the profit-maximizing firm's pricing in mixed duopoly. At low degrees of gamma, the cooperative's influence on the profit-maximizing firm's price is negligible since consumers have a high level of taste for variety. The two goods are close to forming their own markets due to consumers' low willingness to substitute. The pricing decision of the profit-maximizing firm in both duopoly settings is very similar because the cooperative's pricing has little influence. Thus there is little difference between prices in the two settings. At greater degrees of gamma high substitutability itself enforces a low price difference in both settings. When gamma approaches beta the goods become closer to perfect substitutes, which pushes prices to

---

[4] $\frac{1}{1.7} = 0.59$

[5] Recall assumption that $\beta > \gamma$

marginal cost. Thus, the effect of the cooperative is greatest at moderate levels of substitutability when it can influence its competitor's price by placing further downward pressure on it.

I now turn my analysis to quantities.

Quantities in normal duopoly

$$y'(p_{x'}^N, p_{y'}^N) = \frac{\beta}{(\beta^2 - \gamma^2)} \frac{2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma}{(4\beta^2 - \gamma^2)}$$
$$x'(p_{x'}^N, p_{y'}^N) = \frac{\beta}{(\beta^2 - \gamma^2)} \frac{2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma}{(4\beta^2 - \gamma^2)} \qquad (32)$$

Quantities in mixed duopoly

$$y(p_x^N, p_y^N) = \frac{\frac{1}{2}(\beta\alpha - \alpha\gamma)}{\beta^2 - \gamma^2}$$
$$x(p_x^N, p_y^N) = \frac{\beta\alpha - \frac{1}{2}\alpha\gamma - \frac{1}{2\beta}\alpha\gamma^2}{\beta^2 - \gamma^2} \qquad (33)$$

First I analyze the quantity of the good sold by the cooperative and then that sold by the profit-maximizing firm.

$$x(p_x^N, p_y^N) - x'(p_{x'}^N, p_{y'}^N) = \frac{\alpha}{\beta(2\beta - \gamma)(\beta + \gamma)} \left[(\beta + \frac{1}{\sqrt{2}}\gamma)(\beta - \overbrace{\frac{1}{\sqrt{2}}}^{<1}\gamma) \right] \qquad (34)$$

Because all factors in brackets are positive when $\beta > \gamma$, the consumed quantity of the cooperative's good x is always higher in the mixed than in the normal duopoly. This follows from the lower price of the cooperative's good in mixed duopoly compared to the Nash equilibrium price in normal duopoly.

$$y(p_x^N, p_y^N) - y'(p_{x'}^N, p_{y'}^N) = -\frac{\alpha\gamma}{2(2\beta - \gamma)(\beta + \gamma)} \qquad (35)$$

Because $\beta > \gamma$ the above difference between quantities of the profit-maximizing firm's good is always negative. Thus consumption of the profit-maximizing firm’s good y is always higher in normal than in mixed duopoly.

The difference in total quantity is given by the sum of the differences.

$$\begin{aligned} & x(p_x^N, p_y^N) + y(p_x^N, p_y^N) - (x'(p_{x'}^N, p_{y'}^N) + y'(p_{x'}^N, p_{y'}^N)) \\ & = x(p_x^N, p_y^N) - x'(p_{x'}^N, p_{y'}^N) + y(p_x^N, p_y^N) - y'(p_{x'}^N, p_{y'}^N) \quad (36) \\ & = \frac{\alpha(2\beta+\gamma)(\beta-\gamma)}{2\beta(2\beta-\gamma)(\beta+\gamma)} > 0 \end{aligned}$$

Total quantity consumed is always higher in mixed than normal duopoly. Next I examine how quantity differences behave with respect to gamma.

$$\frac{\partial(x(p_x^N, p_y^N) - x'(p_{x'}^N, p_{y'}^N))}{\partial\gamma} = \frac{\partial(y(p_x^N, p_y^N) - y'(p_{x'}^N, p_{y'}^N))}{\partial\gamma} = -\frac{\alpha(2\beta^2+\gamma^2)}{2(2\beta-\gamma)^2(\beta+\gamma)^2} \quad (37)\ \&\ (38)$$

As gamma approaches beta (complete substitutability) the difference between quantities decreases as both settings approach perfect markets. This is also a reflection of how demand behaves differently in the duopolies. In mixed duopoly, quantity is decreasing in gamma whereas in normal duopoly the opposite is true. Higher levels of gamma drive down the original difference in quantities.

6.2. Utility and profit

Consumer utility and firm profit are the most important variables of the two duopoly settings. Next I will display the difference between these variables in normal and mixed duopoly and how this difference behaves with respect to gamma.

Utility in normal duopoly

$$U_{normal} = \frac{\alpha^2\beta^2}{(\beta^2-\gamma^2)^2}\left[\frac{2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)}{(4\beta^2-\gamma^2)^2} - \frac{(\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)^2}{(4\beta^2-\gamma^2)^2})\right] \quad (39)$$

Utility in mixed duopoly

$$U_{mixed} = \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\begin{bmatrix} +\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta-\gamma)^2(\frac{\beta^2+4\beta\gamma+2\gamma^2}{8\beta}) \end{bmatrix} \quad (40)$$

Difference between utilities

$$U_{mixed} - U_{normal} = \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(\beta^2-\gamma^2+\beta^2) \end{bmatrix} > 0 \quad (41)$$

Somewhat unsurprisingly, consumer utility is always higher in mixed duopoly than in normal duopoly. This is due to the fact that in mixed duopoly the pricing power of the profit-maximizing firm is reduced. Consumers can always resort to consuming the cooperative's goods at marginal cost. Because the cooperative is insensitive to the profit-maximizing firm's price changes, the profit-maximizing firm has less room for maintaining a higher price. In normal duopoly with strategic complements the two firms could support higher prices even without explicit collusion.

It is also interesting to employ comparative statics to see how the difference in utilities reacts to the substitutability of goods.

$$\frac{\partial(U_{mixed}-U_{normal})}{\partial\gamma} = -\frac{\alpha^2}{4(\beta+\gamma)^2(2\beta-\gamma)^3}\left[8\beta^3-\gamma^3+6\beta\gamma^2\right] < 0 \quad (42)$$

Although consumer utility is always higher in mixed duopoly than in normal, the above equation shows that the utility difference is decreasing with respect to gamma. This is an obvious result if one recalls that utility in mixed duopoly is decreasing with respect to gamma but increasing in normal duopoly.

Next I take a look at how profits of the profit-maximizing firm compare in the two settings.

Profit in normal duopoly

$$\pi_{y'}(p_{y'}^{N},y')=\frac{\alpha^{2}\beta}{(\beta^{2}-\gamma^{2})(4\beta^{2}-\gamma^{2})^{2}}(2\beta^{2}-\beta\gamma-\gamma^{2})^{2} \qquad (43)$$

Profit in mixed duopoly

$$\pi(p_{y},y)=\frac{\alpha^{2}\beta}{(\beta^{2}-\gamma^{2})(4\beta^{2}-\gamma^{2})}\frac{(\beta-\gamma)^{2}(4\beta^{2}-\gamma^{2})}{4\beta^{2}} \qquad (44)$$

Difference between profits

$$\pi_{y'}(p_{y'}^{N},y')-\pi(p_{y},y)=\frac{\gamma\alpha^{2}(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^{2}}\left[4\beta-\gamma\right] \qquad (45)$$

Since $\beta>\gamma$, $4\beta-\gamma$ must also be positive. Thus the profit-maximizing firm's profit is lower in mixed than normal duopoly. The same discussion that applied to higher utility for the consumer applies to lower profit for the firm. Lower pricing power in mixed duopoly limits the price which the profit-maximizing firm can sustain.

I also employ comparative statics to see how the difference in profits responds to the level of gamma.

$$\frac{\partial(\pi_{y'}(p_{y'}^N, y') - \pi(p_y, y))}{\partial\gamma}$$

$$= \frac{2\alpha^2}{(\beta+\gamma)^2(2\beta-\gamma)^3}\left[(\beta - \overbrace{\frac{1}{1-\frac{1}{\sqrt{5}}}}^{\approx 1.8}\gamma)\overbrace{(\beta^2 - \frac{1-\frac{2}{\sqrt{5}}}{1-\frac{1}{\sqrt{5}}}\beta\gamma + \frac{\sqrt{5}-1}{4\sqrt{5}}\gamma^2)}^{>0}\right] \quad (46)$$

Comparative statics show that the derivative of the difference between profits is positive with low values (in this case when $\approx 0.55\beta > \gamma$)[6] and negative with high values of gamma relative to beta ($\approx 0.55\beta < \gamma$). Ignoring the nature of gamma and focusing only on the mechanics of the model, the difference between profits is first increasing and then, after reaching its maximum at a given threshold, decreasing with respect to gamma. Despite the change in the derivative's sign, the profit-maximizing firm's profit in normal duopoly is still always higher than in mixed duopoly. The initial increasing difference comes from the increase in the cooperative's effect on the profit-maximizing firm's price as discussed before in the case of price (31). When gamma is close to zero the two market settings are similar to each other because the profit-maximizing firm is shielded from the cooperative's pricing effect by low substitutability. The decreasing difference is explained by the diminishing pricing power as consumers are more willing to substitute one good for the other. As gamma approaches beta, the two market settings become again more alike.

6.3. Total surplus

Total surplus (*TS*) is the sum of firm profit and consumer utility. In mixed duopoly it is the sum the representative cooperative member's profit and the profit-maximizing firm's profit. In the normal duopoly case it is the sum of the two firms' profits and consumer utility. The previously calculated differences between variables in the two duopolies can be used to calculate the difference in total surplus. I sustain the simplification of $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $c_x = c_y = 0$.

---

[6] $\frac{1}{1.8} = 0.55$

$$\begin{aligned}\Delta TS &= TS_{mixed} - TS_{normal}\\ &= (U_{mixed} + \pi_{mixed}) - (2\pi_{normal} + U_{normal})\\ &= (U_{mixed} - U_{normal}) + (\pi_{mixed} - \pi_{normal}) - \pi_{normal}\end{aligned} \tag{47}$$

$$TS_{mixed} - TS_{normal} = \frac{\alpha^2(\beta-\gamma)}{2\beta(\beta+\gamma)(2\beta-\gamma)^2}\left[\underbrace{(\beta + \frac{\sqrt{2}-1}{2}\gamma)}_{>0}(\beta - \underbrace{\frac{\sqrt{2}+1}{2}}_{\approx 1.207}\gamma)\right]$$

From this, one can see that total surplus can be lower in mixed duopoly than in normal duopoly. Only when gamma is below a certain threshold (in this case $\approx 0.83\beta > \gamma$)[7] is mixed duopoly preferable from a total surplus point of view. To see why, one must return to the individual components. The difference between utilities is decreasing with respect to gamma due to decreasing utility from variety in mixed duopoly, which is not compensated by a strong enough lower price effect as it is in normal duopoly. Although consumer utility is always higher in mixed duopoly, the fact that it is decreasing means that at some point the higher profit-maximizing firm profit in normal duopoly overshadows higher consumer utility in mixed duopoly. With this model the tipping point happens to be when $0.83\beta = \gamma$.

I also formulate the derivative of the difference in total surpluses to see how the difference evolves with respect to gamma.

$$\frac{\partial(TS_{mixed} - TS_{normal})}{\partial\gamma} = -2\alpha^2 \frac{(\beta - 1.086\gamma)\overbrace{(\beta^2 + 0.086\beta\gamma + .345\gamma^2)}^{>0}}{(\beta+\gamma)^2(2\beta-\gamma)^3} \tag{48}$$

Taking the derivative of the difference in total surplus with respect to gamma shows that the difference is decreasing for lower ranges of gamma. When gamma approaches beta the difference in surpluses is reversed. This is a natural state of affairs in the light of the earlier finding of how normal duopoly total surplus can be more than mixed duopoly total surplus.

---

[7] $\frac{1}{1.207} = 0.83$

7. Extensions to the basic model

In this section I will consider two extensions to the Bertrand competition model that I have presented before. First I change the market setting from Bertrand to Stackelberg and see how this change affects price. In the second extension part of the population is exogenously excluded from the cooperative, which makes an analysis of sales to non-members possible.

7.1. Stackelberg competition

A characteristic of Bertrand competition is that agents set prices simultaneously. However, this is not the only way to model firm behavior. In fact, firms seldom take action simultaneously. It is conceivable that one firm has gained market leadership through previous success or earlier market entry. Whatever the reason, this firm will be the market leader whose actions are followed by others. Stackelberg competition is a framework for analyzing this kind of setting where one firm leads and others react. In Finland, S-group has the greatest market share with a 40 per cent of the market. Due to its large market size, the cooperative could have achieved status of Stackelberg leader. Moreover, it is possible that K-group, who is the second largest player in the market, follows the decisions made by S-group and only then decides on its own course of action.

In an extension to the basic Bertrand model I will now examine Stackelberg competition in which the cooperative sets prices first and the profit-maximizing firm sets prices next. It will be interesting to see how the behavior of the cooperative firm changes in Stackelberg competition. In this analysis I will assume $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $c_x = c_y = 0$. To solve Nash equilibrium, I will utilize backward induction and first solve the follower's optimal response and thereafter the leader's action.

$$\max_{p_y} \pi(p_y, y) = \frac{\beta\alpha - \alpha\gamma}{\delta} p_y - \frac{\beta}{\delta} p_y^2 + \frac{\gamma}{\delta} p_x p_y \quad // \beta^2 - \gamma^2 = \delta \qquad (49)$$

The reaction curve of the profit-maximizing firm is the same as earlier (7):

$$F.O.C.(\frac{d}{dp_y}) \Rightarrow RC_y = p_y(p_x) = \frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x \quad (50)$$

The Stackelberg leader optimizes her price taking into account the above optimal reaction of the follower.

$$\max_{p_x} = \alpha x + \alpha y - (\beta x^2 + 2\gamma yx + \beta y^2)/2 - p_y y$$
$$\Rightarrow p_x = \frac{\alpha\gamma(\beta-\gamma)}{(\gamma+2\beta)\underbrace{(\gamma-2\beta)}_{\beta>\gamma\to<0}} < 0 \quad (51)$$

An interesting finding from this analysis is that at the Nash equilibrium of Stackelberg competition, the cooperative firm prices below marginal cost. The reason for this is clear. When the cooperative is the Stackelberg leader it pays off to commit to a price below marginal cost because by doing so cooperative members can reach a higher level of utility.

$$p_y = \frac{\alpha(\beta-\gamma)}{\beta}\left[\frac{2\beta^2-\gamma^2}{(\gamma+2\beta)(2\beta-\gamma)}\right] > 0 \quad (52)$$

However, the cooperative cannot induce the profit-maximizing firm to price below costs but can only diminish its profit. Comparing the price of the profit-maximizing firm's good in this Stackelberg setting to that of the Bertrand setting (10) reveals that the price of the firm is indeed lower in Bertrand.

Price of the profit-maximizing firm's good

$$\overbrace{\frac{\alpha(\beta-\gamma)}{\beta}\left[\frac{2\beta^2-\gamma^2}{(\gamma+2\beta)(2\beta-\gamma)}\right]}^{Stackelberg} < \overbrace{\frac{\alpha(\beta-\gamma)}{2\beta}}^{Bertrand} \qquad (53)^{8}$$

As a result of the above pricing decision consumers are necessarily better off but the profit-maximizing firm's profit is squeezed further. Is this kind of behavior sustainable? From the consumer's point of view she would need to recapitalize the cooperative because it would sell under marginal costs. However, this would be offset by the ability to influence the profit-maximizing firm's price to a lower level.

7.2. Outside sales

Previously all consumers were assumed to be members of the cooperative. From a Finnish point of view, this restriction leaves out an important part of reality because all consumers not are part of S-group but nevertheless buy goods from it. Now I will consider a model which takes into account the cooperative's sales to non-members. In this setting an exogenous share of the population is not part of the cooperative. Consumers could decide not to join the cooperative for a number of reasons. Some might have ideological reasons for not joining, some might find the initial cash outlay to pay the membership fee too high and others may simply be unaware of how and why to join.

A fraction of the population *N*, denoted by *a*, are members of the cooperative. The remainder are not members and make up *(1-a)N* of the population. The market setting is as before and includes a profit-maximizing firm and a cooperative. The two firms compete with horizontally differentiated goods in Bertrand competition by simultaneously setting prices. There are two goods on the market. Good *y* is the good sold by the profit-maximizing firm and *x* is sold by the cooperative. I use subscript *m* for the goods demanded by cooperative members and *r* for the non-members. Marginal costs are set to zero and thus prices are to be interpreted as mark-ups. The price of the profit-maximizing firm's goods is p_y and the price

[8] For proof see equations appendix

of the cooperative's good is p_x. The cooperative sells to non-members above marginal cost which departs from its pricing to members to whom it sells at marginal cost.

The profit-maximizing firm maximizes profit by optimizing price. Demand for its good comes from two sources: cooperative members and non-members. Thus total profit π consists of revenue from both groups. Likewise, the cooperative faces two demands: demand from its members and that from non-members. Unlike the profit-maximizing firm, the cooperative has two different prices. Its price for members is zero but it charges p_x for sales to non-members. Cooperative total profit is depicted as $\prod_{coop}$ and per member profit as π_{coop}. The utility functions for the representative member and non-member are identical and as before (1). For simplicity, I assume $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $c_x = c_y = 0$.

First I derive the profit functions and the net utility functions.

$$\begin{aligned} &\prod\nolimits_{coop} = aNx_m * 0 + (1-a)Nx_r p_x \\ &\Leftrightarrow \prod\nolimits_{coop} = (1-a)Nx_r p_x \\ &\pi_{coop} = \frac{\prod_{coop}}{aN} = \frac{(1-a)}{a} x_r p_x \end{aligned} \tag{54}$$

$$\max_{p_y} \pi = p_y y_m + p_y y_r$$

$$\max_{p_x} U_m = \alpha x_m + \alpha y_m - \tfrac{1}{2}(\beta x_m^2 + 2\gamma x_m y_m + \beta y_m^2) - x_m * 0 - p_y y_m + \overbrace{\frac{(1-a)}{a} x_r p_x}^{\pi_{coop}} \tag{55}$$

$$U_r = \alpha x_r + \alpha y_r - \tfrac{1}{2}(\beta x_r^2 + 2\gamma x_r y_r + \beta y_r^2) - p_x x_r - p_y y_r$$

Note that the representative member's utility function includes her share of the cooperative's profit from sales to non-members.

Next I solve direct demands y_m, y_r, x_m and x_r.

$$Members' \quad demand$$

$$y_m = \frac{\alpha\beta - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y \qquad (56)$$
$$x_m = \frac{\alpha\beta - \alpha\gamma}{\beta^2 - \gamma^2} + \frac{\gamma}{\beta^2 - \gamma^2} p_y$$

$$Non-members' \quad demand$$

$$y_r = \frac{\beta\alpha - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y + \frac{\gamma}{\beta^2 - \gamma^2} p_x \qquad (57)$$
$$x_r = \frac{\beta\alpha - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_x + \frac{\gamma}{\beta^2 - \gamma^2} p_y$$

The profit-maximizing firm and the cooperative face these four demand schedules. Next I will derive their reaction curves starting with the price of the profit-maximizing firm's good.

$$p_y(p_x) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \overbrace{\frac{\gamma}{4\beta}}^{slope} p_x \qquad (58)$$

The cooperative's reaction for its price to non-members is as follows.

$$p_x(p_y) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \overbrace{\frac{\gamma}{2\beta}}^{slope} p_y \qquad (59)$$

The slope of the cooperative's reaction curve is twice that of the profit-maximizing firm's. Also, the cooperative's reaction curve is the same as the profit-maximizing firm's when cooperative sales to non-members were prohibited (7). The difference in reaction curves stems from the cooperative's ability to apply dual pricing. It sells goods at marginal cost to members but sells at p_x to non-members. However, the profit-maximizing firm is bound in this model to one price for both groups. This diminishes the effect of the cooperative's price on the price of the profit-maximizing firm thus leading to a lower slope in its reaction curve

Next I derive Nash equilibrium prices.

$$p_x^N = \frac{2\alpha(\beta-\gamma)(2\beta+\gamma)}{8\beta^2-\gamma^2}$$
$$p_y^N = \frac{2\alpha(\beta-\gamma)(2\beta+\frac{1}{2}\gamma)}{8\beta^2-\gamma^2} \qquad (60)$$

Nash equilibrium prices are asymmetric due to different reaction curves. The cooperative is able to price its good higher than the profit-maximizing firm. As discussed above, this is because some consumers are cooperative members and some are not. A hike in the cooperative's price reduces demand from non-members only while a corresponding move on the profit-maximizing firm's price reduces demand from both cooperative members and non-members.

It is interesting to compare how these Nash equilibrium prices compare to those of normal (19) or mixed duopoly (10) with only sales to members.

Comparison of the profit-maximizing firm's pricing

$$\overbrace{\frac{2\alpha(\beta-\gamma)(\beta+\frac{1}{2}\gamma)}{4\beta^2-\gamma^2}}^{\text{normal duopoly}} > \overbrace{\frac{2\alpha(\beta-\gamma)(2\beta+\frac{1}{2}\gamma)}{8\beta^2-\gamma^2}}^{\text{mixed duopoly: cooperative sales also to non-members}} > \overbrace{\frac{\beta\alpha-\alpha\gamma}{2\beta}}^{\text{mixed duopoly: cooperative sales only to members}} \qquad (61\ \&\ 62)^9$$

Comparison to mixed and normal duopoly reveals that the price of the profit-maximizing firm's good is highest in normal duopoly. If some consumers are not part of the cooperative and sales are allowed to them, the profit-maximizing firm's price is higher than if all consumers would be members of the cooperative.

---

[9] For proof see equations appendix

Comparison of the cooperative's pricing

$$\overbrace{\frac{2\alpha(\beta-\gamma)(2\beta+\gamma)}{8\beta^2-\gamma^2}}^{\substack{\textit{mixed duopoly:}\\ \textit{cooperative sales also to non-members}}} > \overbrace{0}^{\substack{\textit{mixed duopoly:}\\ \textit{cooperative sales only to members}}} \qquad (63)$$

The price of the cooperative's good is naturally higher in this setting since when everybody is a member, the cooperative prices at marginal cost.

8. Relating S-group behavior to the model

The purpose of this section is to take a step back from modeling and reflect on how it fits Finnish reality. First I will look at the role of S-group and consider how its behavior fits that of a theoretical cooperative. Second I will touch on the subject of the cooperative's saving and investment decision. Third I will look at how the cooperative disburses excess profit. Last I review some recent evidence of S-group's pricing.

The institutions and aims of S-group are what one might expect from a cooperative. There is no reason to suspect that elected members and the management they incentivize would not have maximizing individual consumer surplus as their goal when making decisions. Indeed, this point of view is echoed in S-group's stated purpose of "producing services and benefits for cooperative members" (21, S-group, 2009a). This supports the model framework in which the cooperative maximizes individual consumer utility.

Another outcome of the model is that the profit of the cooperative should always be zero. One might conclude from S-group's annual statements that a decision to withhold profit would mean that the cooperative is acting against what the model would predict. However, the model is static and thus disregards saving and investment decisions that are central to any business. Determining what level of investment is optimal, e.g. opening new stores, is next to impossible ex ante. Additionally, what might not look like a profitable venture for a profit-maximizing firm might just be worthwhile for a cooperative aiming for marginal cost price. The model in this thesis fails to take into account a dynamic setting of intertemporal saving and investment behavior, which would be an interesting field for future research.

S-group uses a membership card to record purchases and to identify members for members-only benefits and discounts. Based on the use of the membership card, the cooperative calculates the member's share of the cooperative's disbursable profit. To simplify the process, each member cooperative has a bonus schedule which dictates how much a given member receives from the cooperative each month. The amount returned, called bonus, depends on the amount the member has spent on the cooperative. This is compatible with the earlier introduced Roschdale principles. In addition to receiving a bonus on their purchases, the cooperative can decide to disburse accumulated profit by paying interest on the member fee.

The regional cooperative of the Helsinki area paid a ten percent interest on each member's membership fee from its 2008 profit (HOK-Elanto, 2009).

The percentage rate determining how much of purchases is returned to members as bonus is increasing with respect to purchases. For example, the cooperative in the Helsinki area, HOK-Elanto, returns three per cent of purchases if a member's purchases are between 300 euros and 400 euros in a given month. The rate of return increases with purchases so that if a member spends over 1500 euros a month on the cooperative, she is returned five per cent of the spent sum (HOK-Elanto, 2010). The non-linear bonus payment structure used by S-group differs from the pure linear scheme I utilized in my model. It also differs from what one might assume from a cooperative. Having a progressive bonus scheme does not seem to be in the spirit of the cooperative ideal because it is against equal treatment of members. In essence, it is a way to charge different prices from different types of consumers and attract members to centralize their purchases into the cooperative. In other words S-group is performing (second degree) price discrimination towards its members. Investigating what the welfare effects of this behavior is left unstudied but would be an interesting field to pursue.

S-group's competitors have also adopted loyalty card programs. For example K-group's "Plussa" card program is widespread. Like S-group, K-group rewards loyalty by paying a bonus on the basis of purchases. Also like its cooperative competitor, K-group employs a non-linear bonus payment structure. K-group is a profit-maximizing firm and thus does not return profit to its customers. This is the likely reason why its customers receive less bonus from purchases than if they would have used the same money in S-group stores. For example: purchasing 400 euros worth of goods yields a return of 10 euros from S-group but only seven euros from K-group. Membership or loyalty cards can also have other uses for a firm. One important use is data collection. Retailers can use data collected from card use to analyze their consumers' purchasing behavior.

Finland's VAT on food was decreased in October 2009 from 17 to 12 per cent. To examine the effect of the VAT decrease, the National Consumer Research Centre (2010) conducted a study on food prices before and after the VAT decrease. The study found that the price difference between the retail chains had decreased from 2008 to 2009. Moreover, the researchers state that "it cannot be said with a high level of confidence that price differences between retailer chains are statistically significant" (National Consumer Research Centre, 2010, 39). In the study, data revealed that Lidl had the lowest prices. However, Lidl operates as a hard discounter involving a different pricing, variety and location strategy. This makes direct comparison to other players in the Finnish market difficult.

The cheapest normal retailer was S-group followed by K-group. Price data used in the sample is normal price and does not take into account price discounts or bonus payments (National Consumer Research Centre, 2010, 10). Since K-group's pricing is more geared to price discounts and S-group's to rewarding bonuses, including these factors would shed light to the actual price difference between the two players. The price study lends authority to the basic model's prediction of pricing behavior. S-group is pricing below its duopoly counterpart K-group as suggested and its goal of price leadership is also reflected in S-group's strategy: "The central goal and competitive advantage of S-group is our continuously cheap price for groceries" (S-group, 2009a). I can conclude that S-group is aiming for price leadership as suggested by the basic model and, based on the recent study, is also executing it. However, the model which considered that some consumers were not part of the cooperative predicts that normal prices would be higher in S-group, which is contrary to the study's findings. This highlights that more research in the model and cooperatives is needed.

In summary, the modeled cooperative matches actual S-group behavior in some aspects but not others. S-group's statements back the assumption that the cooperative maximizes individual consumer utility. The static model does not take into account saving and investment behavior which clearly S-group management needs to consider. This distorts the comparison between the model's prediction of zero profit and annual reports. Additionally, S-group's way of paying out excess profit in a non-linear fashion is contrary to what is modeled. Based on a recent study S-group is the cheapest normal grocery retailer although Lidl, a hard discounter, sustains an even lower price. This observation is in tune with the basic model's prediction of price leadership. All of these areas would be interesting directions to further develop this thesis's analysis.

9. Conclusions

The goal of this thesis is to study the grocery retail trade and how a cooperative might affect the market. In this section I will first summarize the findings from modeling a Bertrand duopoly with a profit-maximizing firm and a cooperative (mixed duopoly). I also compare the resulting Nash equilibrium with the Nash equilibrium of two profit-maximizing firms (normal duopoly). I start with price and continue with quantity, profit, utility and total surplus. I also summarize my findings from extending the basic Bertrand model to first Stackelberg competition and second to a setting where all consumers are not members of the cooperative. I will then relate these findings to reality. Thereafter I will touch on some potential policy considerations before closing with some ideas for further research.

Bertrand duopoly with horizontally differentiated goods is normally modeled assuming two profit-maximizing firms. In this thesis one of the duopolists is assumed to be a consumer-owned retail cooperative, which has a profound effect on market equilibrium. The most important characteristic of the resulting market equilibrium is that the price of the cooperative's good is set equal to marginal cost (see equation 8). This is also intuitively attractive. After all, in the basic model's assumptions the cooperative is barred from sell to non-members and thus the effective price to the owner-members must be equal to cost. An important parameter in the model is the willingness of consumers to substitute one good for the other (parameter gamma in the model). The price of the cooperative's good is independent of gamma but the price of the profit-maximizing firm's good is negatively correlated to gamma. The more willing consumers are to substitute goods the lower the profit-maximizing firm's profit.

Comparison between the Nash equilibrium price of normal duopoly and mixed duopoly reveals differences between the two duopoly settings. The price of the cooperative's good in mixed duopoly is naturally lower than the price of either symmetric good in normal duopoly (28). The gap between prices in the different duopoly settings is lower when consumers do not value variety and are ready to substitute one good for the other (29). This is because the market power of the profit-maximizing firms in normal duopoly is the lower the more consumers are ready to substitute one good for the other. The price of the profit-maximizing firm's good in mixed duopoly is always lower than the corresponding price in normal duopoly due to the cooperative’s choice to price at marginal cost (30). However, this price difference is not monotonous but has a local maximum at a value of consumers’ willingness to substitute (31).

An interesting finding is that at low values of willingness to substitute (and high value for variety), the two duopoly settings are actually quite similar regarding the good sold by the profit-maximizing firm. Because goods are not readily substituted, firms operate as near monopolists with respect to their own markets. Only if consumers are ready to substitute the profit-maximizing firm’s good for the cooperative’s good will the cooperative’s marginal cost pricing start to affect the profit-maximizing firm’s pricing and increase the difference between the profit-maximizing firm’s pricing in normal and mixed duopoly. If consumers are almost indifferent to which good they consume, i.e. they do not value variety, the force of competition will drive prices in both settings towards perfect market pricing, pricing at marginal cost, which overshadows the effect of the cooperative’s marginal cost pricing on its competitor’s pricing decision.

The findings on price are of particular interest because there has long been an active debate about whether Finnish grocery prices are high and if yes, why? Some have blamed the duopolists S-group and K-group of using market power to uphold high prices. However, the findings of this thesis would support the view that the current situation with mixed duopoly is in fact beneficial to that of two profit-maximizing firms and thus the adverse effect of a concentrated market structure is decreased due to the cooperative’s unwillingness to use market power.

Like price, consumed quantity is different in the Nash equilibria of the two duopoly settings. It also behaves differently with respect to consumers’ willingness to substitute. Due to the marginal cost pricing of the cooperative good in mixed duopoly, it is no surprise that the consumed quantity of the cooperative’s good is higher in mixed duopoly than consumption of the goods in normal duopoly (34). On the other hand, the consumed quantity of the profit-maximizing firm’s good in mixed duopoly is lower than consumption in normal duopoly (35). The less consumers value variety and the readier they are to substitute one good for the other, the less (more) they consume in mixed (normal) duopoly (15, 24). Thus the higher gamma is the less quantities in the duopoly settings differ (32 & 33).

A characteristic of demand in mixed duopoly is that it is decreasing with respect to consumers' willingness to substitute. While a higher level of substitutability and lower taste for variety has the effect of lowering the price of the non-cooperatively sold good (boosting demand), lower taste for variety obviously means less utility from variety (depressing demand). In normal duopoly and with this particular utility function, the loss of utility from variety is curtailed by the effect of increasing substitutability on price. This force is considerably weaker in mixed duopoly where prices are lower due to the cooperative's choice to price at marginal cost.

The profit-maximizing firm's profit in mixed duopoly is at its highest when consumers are unwilling to substitute one good for the other (16). This is a logical consequence of pricing power due to the consumers' reluctance to substitute and applies to both duopoly settings (16 & 25). The difference between the profit-maximizing firm's profit in mixed duopoly and profit in normal duopoly is not monotonous. This is due to the same reason as with prices. At low values of gamma, i.e. low willingness to substitute and high value for variety, there is little substitution effect between the two goods in both settings and thus the duopoly settings resemble each other. At higher values of gamma, consumers are more willing to substitute goods. In mixed duopoly the profit-maximizing firm is affected by the cooperative's marginal cost pricing, which is the root cause for the difference between profits in the duopoly settings.

An important finding of this thesis is that consumer utility is decreasing with respect to gamma in mixed duopoly (17). This is contrary to what is the case in normal duopoly (26). The reason for this difference to normal duopoly is, as with quantity, consumers value variety. A higher level of gamma means less utility from variety reducing utility. This reduction of utility is not countered by a big enough decrease of price due to the cooperative's already aggressive pricing and thus the decrease in utility from variety is the prevailing force. However, at a given level of gamma utility is always higher in mixed duopoly than in normal duopoly. The difference is decreasing in gamma as an increase in consumers' willingness to substitute one good for the other pushes normal duopoly prices closer to marginal cost thus decreasing the pricing power of the profit-maximizing firms.

In both duopoly settings total surplus is highest at low levels of gamma meaning a low willingness to substitute and high value of variety (18 & 27). In mixed duopoly this is a direct result of both profit and utility being the higher the lower gamma is. In normal duopoly utility is higher at large values of gamma but not high enough to counter reduced firm profit in a setting where consumers readily consume the cheapest good. At low levels of gamma mixed duopoly total surplus is higher than in normal duopoly but is decreasing faster with respect to gamma (48). What follows is that with high values of gamma, normal duopoly exhibits a higher level of total surplus than mixed duopoly (47).

After modeling the market in Bertrand competition with all members of the population as cooperative members I expanded the analysis. I first changed decision-making so that instead of choosing prices simultaneously the cooperative was made to commit to a price first. The resulting Stackelberg competition leads to the cooperative pricing below marginal cost (51). Consequently, the price chosen by the profit-maximizing firm is below what it was in Bertrand competition (52 & 53). Herein lies the rationale for the cooperative’s negative pricing. By precommitting to a negative price it can influence the profit-maximizing firm to price at lower a lower level than what it would if the firms chose simultaneously. In reality the outcome predicted by the Stackelberg model could be sustained. Members of the cooperative would be better off so they should not object to the practice. Also, the cooperative could sustain losses by adjusting the amount it returns to customers. An interesting continuation of research would be to look into whether there are signs that S-group is selling to members below marginal costs (after adjusting for bonus payments).

The second extension to the basic model is made in order to study how sales to cooperative non-members affects equilibrium. I exogenously set a portion of the population to not be members of the cooperative. This results in dual pricing for the cooperative. For members, goods are sold at marginal cost but pricing power is used against the non-members. The profit-maximizing firm uses one price for sales to members and non-members. As a result, the reaction curve of the profit-maximizing firm is less elastic to the price of the cooperative than the reaction curve of the cooperative is to the profit-maximizing firm’s price. Thus the cooperative’s price is higher than the profit-maximizing firm’s price. This is because a price change by the profit-maximizing firm is directed at the whole population whereas the cooperative’s actions only affect cooperative non-members. Nash equilibrium prices are asymmetric with the cooperative having a higher price than the profit-maximizing firm. The profit-maximizing firm’s price is higher than in the earlier model now that the cooperative is allowed to sell to non-members. Whether the model’s predictions hold with K-group and S-group is not investigated in this thesis but could be an interesting research pursuit.

In this thesis I have not used a general model but have rather used a particular utility function and assumed no cost differences between firms. These give rise to restrictions on how results can be applied to real-world considerations. However, if one accepts that the simplified model has some resonance with real world activities I can present some tentative ideas on how a cooperative changes a duopoly. The immediate effect is that cooperative marginal cost pricing reins in the market power of the profit-maximizing firm. Thus, if consumers face a choice between a duopoly of two profit-maximizing firms and a duopoly of a cooperative and a profit-maximizing firm, they would be better off choosing the latter. If one takes the duopoly structure in Finnish grocery trade as given, consumers should be pleased that S-group, due to its cooperative setup, limits its own and K-group's market power. If one takes a total surplus point of view the answer is ambiguous. Either setting of duopoly can deliver higher levels of total surplus. Which duopoly setting delivers a higher level of total surplus depends on how much consumers value good variety and how ready they are to substitute one good for the other.

If one takes mixed duopoly as given, profit, consumer utility and total surplus are all the higher the more consumers value good variety and the less they are willing to substitute one good for the other, i.e. low gamma. Policy makers can influence consumers' willingness to substitute. The most obvious way is zoning. The results of the model mean that S-group and K-group stores should not be discouraged from locating at different locations based on competition concerns. This is nontrivial because the current trend has seen an increase of locating competing hypermarkets of both groups almost opposite of each other, which decreases consumers' choice of locations to shop at. Reasons for this kind of zoning include environmental concerns or a desire to form a given urban framework. However, based on this analysis such practice cannot be defended based on increased competition because due to S-group's assumed marginal cost pricing an increased possibility to substitute shopping from K-group's hypermarket to S-group's hypermarket does not have a strong price effect.

In summary, S-group's cooperative form decreases the adverse effects of duopoly and allows consumers value a high level of variety without giving too much pricing power to the duopoly firms. For these model results to hold S-group must keep its cooperative form. This is important because if policy makers take a relaxed view on the duopoly setting, they must be sure that S-group retains it cooperative form. Although a change in S-group's setting is presently a remote possibility, the Finnish depression in the 1990s caused one cooperative in the grocery retail trade to change into a profit-maximizing firm.

An interesting topic for further research would be to analyze duopoly competition without assuming a particular form for the consumer utility function. The linear demand system used in this thesis has allowed me to explicitly formulate Nash equilibria. However, any conclusions are only indicative and would be strengthened by treating the same problem with a generalized utility function.

Expanding from a static setting to a dynamic model might also yield interesting findings. The model used in this thesis only looked at one period and disregarded any investment and saving decisions. However, because a profit-maximizing firm and cooperative firm maximize different outcomes one could expect them to make investment decisions differently. S-group might be content with breaking even on a given investment whereas K-group would need to pass some hurdle rate on its return on investment. This would necessarily cause differences in how the two expand.

An analysis of the institutions of cooperatives in general and S-group in particular would be interesting to read. Looking into how S-group's institutions perform from a cost efficiency point of view but also in channeling members' desire into S-group decision making might be worth looking into. Comparing a cooperative's structure to that of a publicly listed firm (maybe as a case study between K-group and S-group) could also be productive.

Investigating different cost structures and their effects on equilibrium would also be an interesting undertaking. One reason for the profit-maximizing firm's prevalence is that it is generally so effective compared to other forms of organization. However, in this thesis I have assumed no differences in cost structures. This is warranted because S-group does not seem to be any more cost inefficient than K-group. It might even be that the opposite is true. In fact, it might be interesting to see how equilibrium changes in the face of a cooperative that is more efficient than its profit-maximizing counterpart. This relates to a more general question. Marginal cost pricing seems to be attainable not only in a perfect market setting but also when a firm is organized as a cooperative. Generally, I would assume that the drawbacks of a cooperative structure involve not only cost inefficiencies but also difficulties in formulating a coherent strategy. After all, defining goals for a cooperative is more difficult than simply asking for profit maximization. However, the example of S-group shows that a cooperative structure can succeed in grocery trade. From a Finnish consumer's point of view this is fortunate since the cooperative structure lessens the adverse effects of market concentration.

10. Literature

Aalto-Setälä, Ville (2002), “The effect of concentration and market power on food prices: evidence from Finland”, *Journal of Retailing*, Vol. 70, No. 3 (Aut. 2002), pp. 207-216

Anderson, Richard K., Porter, Philip K. and Maurice, S. Charles (1979), “The Economics of Consumer-Managed Firms”, *Southern Economic Journal*, Vol. 46, No. 1 (Jul. 1979), pp. 119-130

Bulow, Jermery I., Geanakoplos, John D. and Klemperer, Paul. D. (1985), “Multimarket Oligopoly: Strategic Substitutes and Complements”, *The Journal of Political Economy*, Vol. 93, No. 3 (Jun. 1985), pp. 488-511

Co-operatives Act 1488/2001 (2009), Finnish legislation, http://www.finlex.fi/en/laki/kaannokset/2001/en20011488.pdf, retrieved 31.3.2010

Dixit, Avinash K. and Stiglitz, Joseph E. (1977), “Monopolistic Competition and Optimum Product Diversity”, *The American Economic Review*, Vol. 67, Issue 3 (Jun. 1977), pp. 297-308

Einarsson, Ágúst (2008), “The retail sector in the Nordic countries: A description of the differences, similarities and uniqueness in the global market”, *Journal of Retailing and Consumer Services*, Vol. 15, No. 6 (Nov. 2008), pp. 443-451

Enke, Stephen (1945), “Consumer Cooperatives and Economic Efficiency”, *The American Economic Review*, Vol. 35, No.1 (Mar. 1945), pp. 148-155

Hansmann, Henry (1996), *The Ownership of Enterprise*, Cambridge, Massachusetts

Hansmann, Henry (1999), "Cooperative Firms in Theory and Practice”, *Finnish Journal of Business Economics*, No.4, pp. 387-403

HOK-Elanto (2009), *HOK-Elanto vuosikertomus 2008* (English: Annual report 2008), http://www.hok-elanto.fi/fileadmin/HOK_Elanto_yrityksena/Muut/HOK_Elanto_Vuosikertomus.pdf, retrieved 5.3.2010

HOK-Elanto (2010a), *Welcome as a new member of Helsinki Co-operative Society Elanto*, http://www.hok-elanto.fi/fileadmin/S_Etukortti/Pdf/HOK_Elanto_Welcome_as_a_new_member.pdf, retrieved 5.3.2010

HOK-Elanto (2010b), *HOK-Elannon markkinaosuus nousi 40.9 prosenttiin*, (English: Market Share of HOK-Elanto Increased to 40,9 per cent. Only available in Finnish), http://www.hok-elanto.fi/index.php?id=3837, retrieved 31.3.2010

Holmström, Bengt (1999), "Future of Cooperatives: A Corporate Perspective", *Finnish Journal of Business Economics*, No.4, pp. 404-417

Hotelling, Harold (1929), "Stability in Competition", *The Economic Journal*, Vol. 39, No. 153 (Mar. 1929), pp.41-57

Ireland, Norman J. and Law, Peter J. (1983), "A Cournot-Nash Model of the Consumer Cooperative", *Southern Economic Journal*, Vol. 49, No. 3 (Jan. 1983), pp. 706-716

Jones, Derek C. and Kalmi, Panu (2009), "Co-operatives and Employee Ownership", *International Encyclopedia of Civil Society*, pp. 557-563, Springer New York

K-group annual reports (2000 to 2008), *Keskon vuosi* (only available in Finnish), http://www.kesko.fi/index.asp?id=C8F926B0E7054CFE8893C73D379DAF27, retrieved 16.12.2009

Kalmi, Panu (2006), "The Disappearance of Co-operatives from Economics Textbooks", *Helsinki School of Economics Working Papers*, W-398

Limited Liability Companies Act 624/2006 (2009), Finnish legislation, http://www.finlex.fi/en/laki/kaannokset/2006/en20060624.pdf , retrieved 31.3.2010

National Consumer Research Centre (2010), *Elintarvikkeiden alv-alennus ja hintakehitys*, (English: The VAT reduction and the price development of foodstuffs. Only available in Finnish), http://www.kuluttajatutkimuskeskus.fi/files/5433/2010_01_julkaisu_alv.pdf, retrieved 31.3.2010

Peltola, Heikki (2009), “Kauppakisan voittajat ja voitetut” (English: The winners and losers of grocery competition. Only available in Finnish), *Kaupasta kaikille*, Kuluttajatutkimuksen kirjoja 5, pp. 37-48, Helsinki 2009

Päivittäistavarakauppa ry (2009), *Päivittäistavarakauppa 2009-2010* (English: Grocery trade 2009-2010. Only available in Finnish), http://www.pty.fi/fileadmin/pty_tiedostot/ Julkaisut/47929_PTY_vuosikertomus.pdf, retrieved 21.12.2009

Seppänen, Esko (1993), *Punapääoman romahdus* (English: The Collapse of Red Capital. Only available in Finnish), WSOY, Porvoo

Singh, Nirvikar and Vives, Xavier (1984), ”Price and quantity competition in a differentiated duopoly”, *Rand Journal of Economics*, Vol. 15, No. 4 (Winter 1984), pp. 546-554

S-group (2007), *SOK-yhtymän hallintotapa ja johtamisjärjestelmä* (English: SOK corporate governance. Only available in Finnish), http://www.s-kanava.fi/getbinary?siteId=1&src=/ kuvat/skanava/liitetiedostot/2009/viestinta_Corporate_Governance_2007_fi.pdf, retrieved 17.12.2009

S-group (2009a), *SOK-yhtymä: Vuosikertomus 2008* (English: *SOK annual report*. Only available in Finnish), http://www.s-kanava.fi/getbinary?siteId=1&src=/kuvat/skanava/ liitetiedostot/2009/Vuosiraportointi_2009/SOK_Vuosikertomus_08.pdf, retrieved 15.12.2009

S-group (2009b), *S-ryhmä tänään*, (English: *S-group today*. Only available in Finnish), http://mediapalvelu.s-kanava.fi/s-ryhma/aineistopankki/fi_FI/aineistopankki/_files/ 82089551893367828/default/S_ryhma_tanaan_2008_200309.pdf, retrieved 17.12.2009

S-group (2010), website for a regional cooperative, https://www02.s-kanava.net/liityasiakasomistajaksi/LiityAsiakasomistajaksi.do (Only available in Finnish), visited 5.3.2010

Suomen Lähikauppa Oy (2008), *Tilinpäätös 2008*, (English: Financial Statements 2008. Only available in Finnish), http://www.lahikauppa.fi/client-data/file/tilinpaatostiedote_08.pdf, retrieved 5.3.2010

Talma, Arja (2009), *Kesko Oyj Roadshow*, (presentation by CFO to investors in London 20.11.2009),http://www.kesko.fi/modules/upndown/download_upndownfile.asp?id=AB0AC59770D041C786E89D854F757064&itemtype=UPNDOWNFILE, retrieved 17.12.2009

Thompson, David (1994), "Cooperative Principles Then and Now", *The Cooperative Grocer*, No. 43, July-August 1994, available at http://www.cooperativegrocer.coop/articles/index.php?id=158, retrieved 8.3.2010

United States Department of Justice and the Federal Trade Commission (1997), *Horizontal Merger Guidelines*, http://www.justice.gov/atr/public/guidelines/hmg.htm, retrieved 21.12.2009

Vihma, Päivi (2009), *S-Kauppa vyöryy yli Suomen* (article in Talouselämä magazine 18.3.2009), http://www.talouselama.fi/uutiset/article256290.ece?s=l&wtm=talouselama/-18032009#, retrieved 21.12.2009

11. Equations

(4) Equation

$$U(y,x) - p_x x - p_y y$$
$$\Leftrightarrow \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_x x - p_y y$$

$$F.O.C \quad (to \quad solve \quad inverse \quad demands)$$
$$\frac{\partial}{\partial x} = \alpha_1 - \beta_1 x - \gamma y - p_x = 0$$
$$\Leftrightarrow p_x = \alpha_1 - \beta_1 x - \gamma y$$

$$\frac{\partial}{\partial y} = \alpha_2 - \beta_2 y - \gamma x - p_y = 0$$
$$\Leftrightarrow p_y = \alpha_2 - \beta_2 y - \gamma x$$

Solve direct demands

$$p_y = \alpha_2 - \beta_2 y - \gamma x$$
$$\Leftrightarrow x = \frac{\alpha_2 - \beta_2 y - p_y}{\gamma}$$
$$p_x = \alpha_1 - \beta_1 x - \gamma y \; // \; x = \frac{\alpha_2 - \beta_2 y - p_y}{\gamma}$$
$$\Leftrightarrow p_x = \alpha_1 - \beta_1 \left(\frac{\alpha_2 - \beta_2 y - p_y}{\gamma} \right) - \gamma y \; //{*}\, \gamma$$
$$\Leftrightarrow \gamma p_x - \alpha_1 \gamma + \beta_1 \alpha_2 - \beta_1 p_y = \beta_1 \beta_2 y - \gamma^2 y$$

$$\Leftrightarrow y = \frac{\beta_1 \alpha_2 - \alpha_1 \gamma}{\beta_1 \beta_2 - \gamma^2} - \overbrace{\frac{\beta_1}{\beta_1 \beta_2 - \gamma^2}}^{b_2} p_y + \overbrace{\frac{\gamma}{\beta_1 \beta_2 - \gamma^2}}^{d} p_x$$

Similarly

$$x = \frac{\beta_2 \alpha_1 - \alpha_2 \gamma}{\beta_1 \beta_2 - \gamma^2} - \overbrace{\frac{\beta_2}{\beta_1 \beta_2 - \gamma^2}}^{b_1} p_x + \overbrace{\frac{\gamma}{\beta_1 \beta_2 - \gamma^2}}^{d} p_y$$

(5) Equation

$$\max_{p_x} U(x,y) - p_x x - p_y y + \pi \,//\, \pi = x(p_x - c_x)$$
$$\Leftrightarrow \max_{p_x} U(x,y) - p_x x - p_y y + x(p_x - c_x)$$
$$\Leftrightarrow \max_{p_x} U(x,y) - p_y y - c_x x \,//\, U(x,y) = \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2$$
$$\Leftrightarrow \max_{p_x} \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_y y - c_x x$$

(6) Equation n/a

(7) Equation

$$\max_{p_y} \pi(p_y, y) = yp_y - c_y y$$
$$= y(p_y - c_y) \,//\, y = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x$$
$$= \left(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x\right)(p_y - c_y) \,//\, \beta_1\beta_2 - \gamma^2 = \delta > 0$$
$$= \left(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\delta} - \frac{\beta_1}{\delta} p_y + \frac{\gamma}{\delta} p_x\right)(p_y - c_y)$$
$$F.O.C.\left(\frac{d}{dp_y}\right)$$
$$\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\delta} - \frac{\beta_1}{\delta} p_y + \frac{\gamma}{\delta} p_x - \frac{\beta_1}{\delta}(p_y - c_y) = 0 \,//{*}\,\delta$$
$$\Leftrightarrow \beta_1\alpha_2 - \alpha_1\gamma - \beta_1 p_y + \gamma p_x - \beta_1(p_y - c_y) = 0$$
$$\Leftrightarrow 2\beta_1 p_y = \beta_1\alpha_2 - \alpha_1\gamma + \gamma p_x + \beta_1 c_y$$
$$\Leftrightarrow p_y = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{\gamma}{2\beta_1} p_x + \frac{c_y}{2}$$
$$\Rightarrow RC_y = p_y(p_x) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \overbrace{\frac{\gamma}{2\beta_1}}^{slope} p_x$$

(8) Equation

$$\max_{p_x} \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_y y - c_x x$$

$$\beta_1\beta_2 - \gamma^2 = \delta$$
$$\frac{\partial x}{\partial p_x} = -\frac{\beta_2}{\delta}$$
$$\frac{\partial y}{\partial p_x} = \frac{\gamma}{\delta}$$
$$y = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\delta} - \frac{\beta_1}{\delta} p_y + \frac{\gamma}{\delta} p_x$$
$$x = \frac{\beta_2\alpha_1 - \alpha_2\gamma}{\delta} - \frac{\beta_2}{\delta} p_x + \frac{\gamma}{\delta} p_y$$

$F.O.C$

$$\frac{\partial}{\partial p_x} = \alpha_1 \frac{\partial x}{\partial p_x} + \alpha_2 \frac{\partial y}{\partial p_x} - \frac{1}{2}\left[2\beta_1 x \frac{\partial x}{\partial p_x} + 2\gamma (x \frac{\partial y}{\partial p_x} + y \frac{\partial x}{\partial p_x}) + 2\beta_2 y \frac{\partial y}{\partial p_x}\right] - p_y \frac{\partial y}{\partial p_x} - c_x \frac{\partial x}{\partial p_x} = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2}{\delta} + \frac{\alpha_2\gamma}{\delta} - \frac{p_y\gamma}{\delta} + \frac{c_x\beta_2}{\delta}$$
$$-\frac{1}{2}\begin{bmatrix} 2\beta_1(\frac{\beta_2\alpha_1 - \alpha_2\gamma}{\delta} - \frac{\beta_2}{\delta} p_x + \frac{\gamma}{\delta} p_y) * -\frac{\beta_2}{\delta} \\ + 2\gamma((\frac{\beta_2\alpha_1 - \alpha_2\gamma}{\delta} - \frac{\beta_2}{\delta} p_x + \frac{\gamma}{\delta} p_y) * \frac{\gamma}{\delta} - (\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\delta} - \frac{\beta_1}{\delta} p_y + \frac{\gamma}{\delta} p_x) * -\frac{\beta_2}{\delta}) \\ + 2\beta_2(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\delta} - \frac{\beta_1}{\delta} p_y + \frac{\gamma}{\delta} p_x) * \frac{\gamma}{\delta} \end{bmatrix} = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2}{\delta} + \frac{1}{\delta^2}\begin{bmatrix} \beta_1\beta_2(\beta_2\alpha_1 - \alpha_2\gamma - \beta_2 p_x + \gamma p_y) \\ -\gamma^2(\beta_2\alpha_1 - \alpha_2\gamma - \beta_2 p_x + \gamma p_y) \\ -\beta_2\gamma(\beta_1\alpha_2 - \alpha_1\gamma - \beta_1 p_y + \gamma p_x)) \\ +\gamma\beta_2(\beta_1\alpha_2 - \alpha_1\gamma - \beta_1 p_y + \gamma p_x) \end{bmatrix} = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2}{\delta} + \frac{1}{\delta^2}\begin{bmatrix} \beta_1\beta_2^2\alpha_1 - \beta_1\beta_2\alpha_2\gamma - \beta_1\beta_2^2 p_x + \beta_1\beta_2\gamma p_y \\ -\gamma^2\beta_2\alpha_1 + \gamma^3\alpha_2 + \gamma^2\beta_2 p_x - \gamma^3 p_y \\ -\beta_1\beta_2\alpha_2\gamma + \beta_2\alpha_1\gamma^2 + \beta_1\beta_2 p_y\gamma - \beta_2\gamma^2 p_x \\ +\beta_1\beta_2\alpha_2\gamma - \beta_2\gamma^2\alpha_1 - \beta_1\beta_2 p_y\gamma + \beta_2\gamma^2 p_x \end{bmatrix} = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2}{\delta} + \frac{1}{\delta^2}\begin{bmatrix} \beta_1\beta_2^2\alpha_1 - \beta_1\beta_2\alpha_2\gamma - \beta_1\beta_2^2 p_x + \beta_1\beta_2\gamma p_y \\ -\beta_2\gamma^2\alpha_1 + \gamma^3\alpha_2 + \beta_2\gamma^2 p_x - \gamma^3 p_y \end{bmatrix} = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2}{\delta} + \frac{1}{\delta^2}\left[\overbrace{(\beta_1\beta_2 - \gamma^2)}^{\delta} * (\beta_2\alpha_1 - \alpha_2\gamma - \beta_2 p_x + p_y\gamma) \right] = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2}{\delta} + \frac{\delta}{\delta^2}\left[\beta_2\alpha_1 - \alpha_2\gamma - \beta_2 p_x + p_y\gamma\right] = 0$$

$$\Leftrightarrow \frac{-\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2 + \beta_2\alpha_1 - \alpha_2\gamma - \beta_2 p_x + p_y\gamma}{\delta} = 0 // * \delta$$

$$\Leftrightarrow -\alpha_1\beta_2 + \alpha_2\gamma - p_y\gamma + c_x\beta_2 + \beta_2\alpha_1 - \alpha_2\gamma - \beta_2 p_x + p_y\gamma = 0$$

$$\Leftrightarrow c_x\beta_2 - \beta_2 p_x = 0$$

$$\Leftrightarrow p_x = c_x$$

(9) Equation

$$\pi(p_y, y) = y p_y - c_y y$$

$$= y(p_y - c_y) // y = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x$$

$$= \left(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x\right)(p_y - c_y)$$

$$\frac{\partial\pi}{\partial p_x} = \frac{\gamma}{\beta_1\beta_2 - \gamma^2}(p_y - c_y)$$

$$\frac{\partial^2\pi}{\partial p_x \partial p_y} = \frac{\gamma}{\beta_1\beta_2 - \gamma^2} > 0 // \beta_1\beta_2 - \gamma^2 > 0, \gamma > 0$$

(10) Equation

$$RC_y = p_y(p_x) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \overbrace{\frac{\gamma}{2\beta_1}}^{slope} p_x \; // \; P_x = c_x$$

$$\Rightarrow p_y^N = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1} c_x$$

$$RC_x = c_x$$

$$\Rightarrow p_x^N = c_x$$

(11) Equation

$$y(p_x^N, p_y^N) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y^N + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x^N$$

$$= \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma + \beta_1 c_y + \gamma c_x)}{\beta_1\beta_2 - \gamma^2} + \frac{\gamma c_x}{\beta_1\beta_2 - \gamma^2}$$

$$= \frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\beta_1\beta_2 - \gamma^2} - \frac{\frac{1}{2}\beta_1}{\beta_1\beta_2 - \gamma^2} c_y + \frac{\frac{1}{2}\gamma}{\beta_1\beta_2 - \gamma^2} c_x$$

Similarly

$$x(p_x^N, p_y^N) = \frac{\beta_2\alpha_1 - \alpha_2\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_2}{\beta_1\beta_2 - \gamma^2} p_x^N + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_y^N$$

$$= \frac{\beta_2\alpha_1 - \alpha_2\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_2 c_x}{\beta_1\beta_2 - \gamma^2} + \frac{\gamma}{\beta_1\beta_2 - \gamma^2}\left(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1} c_x\right)$$

$$= \frac{\beta_2\alpha_1 - \alpha_2\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_2 c_x}{\beta_1\beta_2 - \gamma^2} + \frac{\frac{1}{2}\gamma\alpha_2 - \frac{1}{2\beta_1}\alpha_1\gamma^2 + \frac{1}{2}\gamma c_y + \gamma^2 \frac{1}{2\beta_1} c_x}{\beta_1\beta_2 - \gamma^2}$$

$$= \frac{\beta_2\alpha_1 - \frac{1}{2}\alpha_2\gamma - \frac{1}{2\beta_1}\alpha_1\gamma^2}{\beta_1\beta_2 - \gamma^2} - \frac{(\beta_2 - \frac{\gamma^2}{2\beta_1})}{\beta_1\beta_2 - \gamma^2} c_x + \frac{\frac{\gamma}{2}}{\beta_1\beta_2 - \gamma^2} c_y$$

(12) Equation

$$\pi(p_y, y) = y(p_y - c_y) // \left[\begin{array}{l} y = \frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\delta} - \frac{\frac{1}{2}\beta_1}{\delta}c_y + \frac{\frac{1}{2}\gamma}{\delta}c_x, \delta = \beta_1\beta_2 - \gamma^2, \\ p_y^N = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1}c_x \end{array}\right]$$

$$\begin{aligned} &= \left(\frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\delta} - \frac{\frac{1}{2}\beta_1}{\delta}c_y + \frac{\frac{1}{2}\gamma}{\delta}c_x\right)\left(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1}c_x - c_y\right) \\ &= \left(\frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\delta} - \frac{\frac{1}{2}\beta_1}{\delta}c_y + \frac{\frac{1}{2}\gamma}{\delta}c_x\right)\left(\frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} - \frac{\beta_1 c_y}{2\beta_1} + \frac{\gamma}{2\beta_1}c_x\right) \\ &= \frac{(\beta_1\alpha_2 - \alpha_1\gamma - \beta_1 c_y + \gamma c_x)(\beta_1\alpha_2 - \alpha_1\gamma - \beta_1 c_y + \gamma c_x)}{4\beta_1\delta} \\ &= \frac{(\beta_1\alpha_2 - \alpha_1\gamma - \beta_1 c_y + \gamma c_x)^2}{4\beta_1\delta} \end{aligned}$$

(13) Equation

$$U = \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_y y - c_x x // \left[\begin{array}{l} x = \frac{\beta_2\alpha_1 - \frac{1}{2}\alpha_2\gamma - \frac{1}{2\beta_1}\alpha_1\gamma^2}{\beta_1\beta_2 - \gamma^2} - \frac{(\beta_2 - \frac{\gamma^2}{2\beta_1})}{\beta_1\beta_2 - \gamma^2}c_x \\ + \frac{\frac{\gamma}{2}}{\beta_1\beta_2 - \gamma^2}c_y \\ y = \frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\beta_1\beta_2 - \gamma^2} - \frac{\frac{1}{2}\beta_1}{\beta_1\beta_2 - \gamma^2}c_y + \frac{\frac{1}{2}\gamma}{\beta_1\beta_2 - \gamma^2}c_x \\ p_y^N = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1}c_x ;\ p_x^N = c_x \\ \delta = \beta_1\beta_2 - \gamma^2 \end{array}\right]$$

$$= (\alpha_1 - c_x)x + (\alpha_2 - p_y)y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2$$

$$\begin{aligned} \Leftrightarrow U &= \frac{(\alpha_1 - c_x)}{\delta}\left[(\beta_2\alpha_1 - \tfrac{1}{2}\alpha_2\gamma - \tfrac{1}{2\beta_1}\alpha_1\gamma^2) - c_x(\beta_2 - \tfrac{\gamma^2}{2\beta_1}) + c_y(\tfrac{\gamma}{2})\right] \\ &+ \frac{\left(\frac{\alpha_2\beta_1 + \alpha_1\gamma}{2\beta_1} - \frac{c_y}{2} - \frac{\gamma}{2\beta_1}c_x\right)}{\delta}\left[\tfrac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma) - \tfrac{1}{2}\beta_1 c_y + \tfrac{1}{2}\gamma c_x\right] \\ &- \frac{1}{2\delta^2}\left[\begin{array}{l} \beta_1\left((\beta_2\alpha_1 - \tfrac{1}{2}\alpha_2\gamma - \tfrac{1}{2\beta_1}\alpha_1\gamma^2) - c_x(\beta_2 - \tfrac{\gamma^2}{2\beta_1}) + c_y(\tfrac{\gamma}{2})\right)^2 \\ + 2\gamma\left((\beta_2\alpha_1 - \tfrac{1}{2}\alpha_2\gamma - \tfrac{1}{2\beta_1}\alpha_1\gamma^2) - c_x(\beta_2 - \tfrac{\gamma^2}{2\beta_1}) + c_y(\tfrac{\gamma}{2})\right)\left(\tfrac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma) - \tfrac{1}{2}\beta c_y + \tfrac{1}{2}\gamma c_x\right) \\ \beta_2\left(\tfrac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma) - \tfrac{1}{2}\beta c_y + \tfrac{1}{2}\gamma c_x\right)^2 \end{array}\right] \end{aligned}$$

(14) Equation n/a

(15) Equation

$$\frac{\partial y(p_x^N, p_y^N)}{\partial \gamma} = -\frac{\alpha}{2(\beta+\gamma)^2}$$

$$x(p_x^N, p_y^N) = \frac{\alpha}{2\beta}\frac{2\beta^2-\beta\gamma-\gamma^2}{(\beta^2-\gamma^2)}$$

$$\frac{\partial x(p_x^N, p_y^N)}{\partial \gamma} = \frac{\alpha}{2\beta}\frac{(-\beta-2\gamma)(\beta^2-\gamma^2)+2\gamma(2\beta^2-\beta\gamma-\gamma^2)}{(\beta^2-\gamma^2)^2}$$

$$= \frac{\alpha}{2\beta}\frac{-\beta^3+\beta\gamma^2-2\beta^2\gamma+2\gamma^3+4\beta^2\gamma-2\beta\gamma^2-2\gamma^3}{(\beta^2-\gamma^2)^2}$$

$$= -\frac{\alpha}{2}\frac{\beta^2-2\beta\gamma+\gamma^2}{(\beta^2-\gamma^2)^2}$$

$$= -\frac{\alpha}{2}\frac{(\beta-\gamma)^2}{(\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= -\frac{\alpha}{2(\beta+\gamma)^2}$$

(16) Equation

$$\pi(p_y, y) = \frac{\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)}$$

$$= \frac{\alpha^2}{4\beta}\frac{(\beta-\gamma)}{(\beta+\gamma)}$$

$$\frac{\partial \pi(p_y, y)}{\partial \gamma} = \frac{\alpha^2}{4\beta}\frac{-(\beta+\gamma)-(\beta-\gamma)}{(\beta+\gamma)^2}$$

$$= \frac{\alpha^2}{4\beta}\frac{-2\beta}{(\beta+\gamma)^2}$$

$$= -\frac{\alpha^2}{2(\beta+\gamma)^2}$$

(17) Equation

$$U = \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\begin{bmatrix} +\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta-\gamma)^2\left(\frac{\beta^2+4\beta\gamma+2\gamma^2}{8\beta}\right) \end{bmatrix}$$

$$= \frac{\alpha^2}{8\beta(\beta^2-\gamma^2)^2}\begin{bmatrix} 4(\beta+\frac{1}{2}\gamma)^2(\beta-\gamma)^2 \\ +(\beta-\gamma)^2(\beta^2+4\beta\gamma+2\gamma^2) \end{bmatrix}$$

$$= \frac{\alpha^2}{8\beta(\beta-\gamma)^2(\beta+\gamma)^2}\begin{bmatrix} 4(\beta+\frac{1}{2}\gamma)^2(\beta-\gamma)^2 \\ +(\beta-\gamma)^2(\beta^2+4\beta\gamma+2\gamma^2) \end{bmatrix}$$

$$= \frac{\alpha^2}{8\beta(\beta+\gamma)^2}\left[4(\beta+\frac{1}{2}\gamma)^2+\beta^2+4\beta\gamma+2\gamma^2\right]$$

$$= \frac{\alpha^2}{8\beta(\beta+\gamma)^2}\left[4(\beta^2+\beta\gamma+\frac{1}{4}\gamma^2)+\beta^2+4\beta\gamma+2\gamma^2\right]$$

$$= \frac{\alpha^2}{8\beta(\beta+\gamma)^2}\left[4\beta^2+4\beta\gamma+\gamma^2+\beta^2+4\beta\gamma+2\gamma^2\right]$$

$$= \frac{\alpha^2}{8\beta(\beta+\gamma)^2}\left[5\beta^2+8\beta\gamma+3\gamma^2\right]$$

$$= \frac{5\alpha^2}{8\beta(\beta+\gamma)^2}\left[(\beta+\gamma)(\beta+\frac{3}{5}\gamma)\right]$$

$$= \frac{\alpha^2(5\beta+3\gamma)}{8\beta(\beta+\gamma)}$$

$$\frac{\partial U}{\partial\gamma} = \frac{\alpha^2}{8\beta}\frac{3(\beta+\gamma)-5\beta-3\gamma}{(\beta+\gamma)^2}$$

$$= \frac{\alpha^2}{8\beta}\frac{-2\beta}{(\beta+\gamma)^2}$$

$$= -\frac{\alpha^2}{4\beta}\frac{\beta}{(\beta+\gamma)^2}$$

$$= -\frac{\alpha^2}{4(\beta+\gamma)^2}$$

(18) Equation n/a

(19) Equation

$$RC_{y'} = p_{y'}(p_{x'}) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1} p_{x'}$$
$$RC_{x'} = p_{x'}(p_{y'}) = \frac{\beta_2\alpha_1 - \alpha_2\gamma}{2\beta_2} + \frac{c_x}{2} + \frac{\gamma}{2\beta_2} p_{y'}$$

$$\Rightarrow p_{y'}(p_{x'}) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1} p_{x'} \, // \, p_{x'}(p_{y'}) = \frac{\beta_2\alpha_1 - \alpha_2\gamma}{2\beta_2} + \frac{c_x}{2} + \frac{\gamma}{2\beta_2} p_{y'}$$
$$\Leftrightarrow p_{y'} = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1}\left(\frac{\beta_2\alpha_1 - \alpha_2\gamma}{2\beta_2} + \frac{c_x}{2} + \frac{\gamma}{2\beta_2} p_{y'}\right)$$
$$\Leftrightarrow p_{y'} = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma(\beta_2\alpha_1 - \alpha_2\gamma)}{4\beta_1\beta_2} + \frac{\gamma c_x}{4\beta_1} + \frac{\gamma^2}{4\beta_1\beta_2} p_{y'}$$
$$\Leftrightarrow p_{y'}(1 - \frac{\gamma^2}{4\beta_1\beta_2}) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma(\beta_2\alpha_1 - \alpha_2\gamma)}{4\beta_1\beta_2} + \frac{\gamma c_x}{4\beta_1}$$
$$\Leftrightarrow p_{y'}(1 - \frac{\gamma^2}{4\beta_1\beta_2}) = \frac{2\beta_2(\beta_1\alpha_2 - \alpha_1\gamma) + \gamma(\beta_2\alpha_1 - \alpha_2\gamma)}{4\beta_1\beta_2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2} c_x$$
$$\Leftrightarrow p_{y'} = \frac{2\beta_2(\beta_1\alpha_2 - \alpha_1\gamma) + \gamma(\beta_2\alpha_1 - \alpha_2\gamma)}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x$$
$$\Leftrightarrow p_{y'} = \frac{2\alpha_2\beta_1\beta_2 - 2\alpha_1\beta_2\gamma + \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x$$

$$\Rightarrow p_{y'}^N = \frac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x$$

Similarly

$$p_{x'}^N = \frac{2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_x + \frac{\beta_1\gamma}{4\beta_1\beta_2 - \gamma^2} c_y$$

(20) Equation

$$y'(p_{x'}^N, p_{y'}^N) = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_{y'}^N + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_{x'}^N \ // \begin{bmatrix} p_{y'}^N = \dfrac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} \\ + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \dfrac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x \\ p_{x'}^N = \dfrac{2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2}{4\beta_1\beta_2 - \gamma^2} \\ + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_x + \dfrac{\beta_1\gamma}{4\beta_1\beta_2 - \gamma^2} c_y \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} \\ - \dfrac{\beta_1}{\beta_1\beta_2 - \gamma^2} \left(\dfrac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \dfrac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x \right) \\ + \dfrac{\gamma}{\beta_1\beta_2 - \gamma^2} \left(\dfrac{2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_x + \dfrac{\beta_1\gamma}{4\beta_1\beta_2 - \gamma^2} c_y \right) \end{bmatrix}$$

$$= \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} + \frac{-2\alpha_2\beta_1^2\beta_2 + \alpha_1\beta_1\beta_2\gamma + \alpha_2\beta_1\gamma^2 - 2\beta_1^2\beta_2 c_y - \beta_1\beta_2\gamma c_x + 2\alpha_1\beta_1\beta_2\gamma - \alpha_2\beta_1\gamma^2 - \alpha_1\gamma^3 + \beta_1\gamma^2 c_y}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)}$$

$$= \frac{4\alpha_2\beta_1^2\beta_2 - \beta_1\alpha_2\gamma^2 - 4\alpha_1\beta_1\beta_2\gamma + \alpha_1\gamma^3}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)}$$

$$+ \frac{-2\alpha_2\beta_1^2\beta_2 + \alpha_1\beta_1\beta_2\gamma + \alpha_2\beta_1\gamma^2 - 2\beta_1^2\beta_2 c_y - \beta_1\beta_2\gamma c_x + 2\alpha_1\beta_1\beta_2\gamma - \alpha_2\beta_1\gamma^2 - \alpha_1\gamma^3 + \beta_1\gamma^2 c_y}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)}$$

$$= \frac{2\alpha_2\beta_1^2\beta_2 - \beta_1\alpha_2\gamma^2 - \alpha_1\beta_1\beta_2\gamma - 2\beta_1^2\beta_2 c_y - \beta_1\beta_2\gamma c_x{}^2 + \beta_1\gamma^2 c_y}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)}$$

$$\Leftrightarrow y'(p_{x'}^N, p_{y'}^N) = \frac{\beta_1}{(\beta_1\beta_2 - \gamma^2)} \frac{2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y}{(4\beta_1\beta_2 - \gamma^2)}$$

Similarly

$$x'(p_{x'}^N, p_{y'}^N) = \frac{\beta_2}{(\beta_1\beta_2 - \gamma^2)} \frac{2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x}{(4\beta_1\beta_2 - \gamma^2)}$$

(21) Equation

$$\pi_{y'}(p_{y'}^{N}, y') = y'(p_{y'}^{N} - c_y) // \begin{bmatrix} y'(p_{x'}^{N}, p_{y'}^{N}) = \dfrac{\beta_1}{(\beta_1\beta_2 - \gamma^2)} \dfrac{2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y}{(4\beta_1\beta_2 - \gamma^2)} \\ p_{y'}^{N} = \dfrac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y + \dfrac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x \end{bmatrix}$$

$$= y'\left(\frac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2 - \gamma^2}{4\beta_1\beta_2 - \gamma^2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x\right)$$

$$= \left(\frac{\beta_1}{\beta_1\beta_2 - \gamma^2} \frac{(2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2) + (-\beta_2\gamma c_x{}^2) - c_y(2\beta_1\beta_2 - \gamma^2)}{4\beta_1\beta_2 - \gamma^2}\right)$$
$$* \left(\frac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \frac{2\beta_1\beta_2 - \gamma^2}{4\beta_1\beta_2 - \gamma^2} c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x\right)$$

$\Leftrightarrow$

$$\pi_{y'}(p_{y'}^{N}, y') =$$

$$\frac{\beta_1}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \begin{pmatrix} (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)^2 \\ + (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)(2\beta_1\beta_2 - \gamma^2) c_y \\ + (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)(\beta_2\gamma) c_x \\ - \beta_2\gamma c_x{}^2 (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2) \\ - \beta_2\gamma c_x{}^2 c_y (2\beta_1\beta_2 - \gamma^2) \\ - c_x^3 \beta_2^2 \gamma^2 \\ - (2\beta_1\beta_2 - \gamma^2) c_y (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2) \\ - c_y^2 (2\beta_1\beta_2 - \gamma^2)^2 \\ - c_x c_y \beta_2\gamma (2\beta_1\beta_2 - \gamma^2) \end{pmatrix}$$

$$= \frac{\beta_1}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \begin{pmatrix} (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)^2 \\ + (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2)(\beta_2\gamma) c_x - \beta_2\gamma c_x{}^2 (2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2) \\ - \beta_2\gamma c_x{}^2 c_y (2\beta_1\beta_2 - \gamma^2) - c_x^3 \beta_2^2 \gamma^2 - c_y^2 (2\beta_1\beta_2 - \gamma^2)^2 - c_x c_y \beta_2\gamma (2\beta_1\beta_2 - \gamma^2) \end{pmatrix}$$

Similarly

$$\pi_{x'}(p_{x'}^{N}, x') =$$

$$\frac{\beta_2}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2} \begin{pmatrix} (2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2)^2 \\ + (2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2)(\beta_1\gamma) c_x - \beta_1\gamma c_x{}^2 (2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2) \\ - \beta_1\gamma c_y{}^2 c_x (2\beta_1\beta_2 - \gamma^2) - c_y^3 \beta_1^2 \gamma^2 - c_x^2 (2\beta_1\beta_2 - \gamma^2)^2 - c_x c_y \beta_1\gamma (2\beta_1\beta_2 - \gamma^2) \end{pmatrix}$$

(22) Equation

$U =$

$$
\alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_y y - p_x x //
\left[
\begin{array}{l}
x'(p_{x'}^N, p_{y'}^N) = \dfrac{\beta_2}{(\beta_1\beta_2 - \gamma^2)} \\
* \dfrac{2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x}{(4\beta_1\beta_2 - \gamma^2)} \\
y'(p_{x'}^N, p_{y'}^N) = \dfrac{\beta_1}{(\beta_1\beta_2 - \gamma^2)} \\
* \dfrac{2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y}{(4\beta_1\beta_2 - \gamma^2)} \\
p_{y'}^N = \dfrac{2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2\gamma - \alpha_2\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_y \\
+ \dfrac{\beta_2\gamma}{4\beta_1\beta_2 - \gamma^2} c_x \\
p_{x'}^N = \dfrac{2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1\gamma - \alpha_1\gamma^2}{4\beta_1\beta_2 - \gamma^2} + \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2 - \gamma^2} c_x \\
+ \dfrac{\beta_1\gamma}{4\beta_1\beta_2 - \gamma^2} c_y
\end{array}
\right]
$$

$$
\begin{aligned}
&= x(\alpha_1 - p_x) + y(\alpha_2 - p_y) - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 \\
&= \frac{x}{4\beta_1\beta_2 - \gamma^2}(4\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - 2\alpha_1\beta_1\beta_2 + \alpha_2\beta_1\gamma + \alpha_1\gamma^2 - 2\beta_1\beta_2 c_x - \beta_1\gamma c_y) \\
&+ \frac{y}{4\beta_1\beta_2 - \gamma^2}(4\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - 2\alpha_2\beta_1\beta_2 + \alpha_1\beta_2\gamma + \alpha_2\gamma^2 - 2\beta_1\beta_2 c_y - \beta_2\gamma c_x) \\
&- (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 \\
&= \frac{\beta_1 x}{4\beta_1\beta_2 - \gamma^2}(2\alpha_1\beta_2 + \alpha_2\gamma - 2\beta_2 c_x - \gamma c_y) \\
&+ \frac{\beta_2 y}{4\beta_1\beta_2 - \gamma^2}(2\alpha_2\beta_1 + \alpha_1\gamma - 2\beta_1 c_y - \gamma c_x) \\
&- (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 \\
&= \frac{\beta_1\beta_2}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2}
\left[
\begin{array}{l}
(2\alpha_1\beta_2 + \alpha_2\gamma - 2\beta_2 c_x - \gamma c_y) \\
*(2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x)
\end{array}
\right] \\
&+ \frac{\beta_1\beta_2}{(\beta_1\beta_2 - \gamma^2)(4\beta_1\beta_2 - \gamma^2)^2}
\left[
\begin{array}{l}
(2\alpha_2\beta_1 + \alpha_1\gamma - 2\beta_1 c_y - \gamma c_x) \\
*(2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y)
\end{array}
\right] \\
&- \frac{\beta_1\beta_2}{2(\beta_1\beta_2 - \gamma^2)^2(4\beta_1\beta_2 - \gamma^2)^2}
\left[
\begin{array}{l}
\beta_2(2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x)^2 \\
+ 2\gamma(2\alpha_1\beta_1\beta_2 - \alpha_1\gamma^2 - \alpha_2\beta_1\gamma - \beta_1\gamma c_y{}^2 + \gamma^2 c_x - 2\beta_1\beta_2 c_x) \\
*(2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y) \\
+ \beta_1(2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y)^2
\end{array}
\right]
\end{aligned}
$$

(23) Equation

$$p_{y'}^{N} = p_{x'}^{N} = \alpha \frac{2\beta^2 - \beta\gamma - \gamma^2}{4\beta^2 - \gamma^2}$$

$$\frac{\partial p_{y'}^{N}}{\partial \gamma} = \frac{\partial p_{x'}^{N}}{\partial \gamma} = \alpha \frac{(-\beta - 2\gamma)(4\beta^2 - \gamma^2) - (2\beta^2 - \beta\gamma - \gamma^2)(-2\gamma)}{(4\beta^2 - \gamma^2)^2}$$

$$= \alpha \frac{2\gamma(2\beta^2 - \beta\gamma - \gamma^2) - (\beta + 2\gamma)(4\beta^2 - \gamma^2)}{(4\beta^2 - \gamma^2)^2}$$

$$= \alpha \frac{4\beta^2\gamma - 2\beta\gamma^2 - 2\gamma^3 - (4\beta^3 - \beta\gamma^2 + 8\beta^2\gamma - 2\gamma^3)}{(4\beta^2 - \gamma^2)^2}$$

$$= \alpha \frac{4\beta^2\gamma - 2\beta\gamma^2 - 2\gamma^3 - 4\beta^3 + \beta\gamma^2 - 8\beta^2\gamma + 2\gamma^3}{(4\beta^2 - \gamma^2)^2}$$

$$= -\frac{\alpha\beta(4\beta\gamma + 4\beta^2 + \gamma^2)}{(4\beta^2 - \gamma^2)^2}$$

$$= -\frac{\alpha\beta(4\beta\gamma + 4\beta^2 + \gamma^2)}{(2\beta + \gamma)^2(2\beta - \gamma)^2}$$

$$= -\frac{\alpha\beta(4\beta^2 + 4\beta\gamma + \gamma^2)}{(4\beta^2 + 4\beta\gamma + \gamma^2)(2\beta - \gamma)^2}$$

$$= -\frac{\alpha\beta}{(2\beta - \gamma)^2}$$

(24) Equation

$$y'(p_{x'}^{N}, p_{y'}^{N}) = x'(p_{x'}^{N}, p_{y'}^{N}) = \frac{\alpha\beta}{(\beta - \gamma)(\beta + \gamma)} \frac{(2\beta + \gamma)(\beta - \gamma)}{(4\beta^2 - \gamma^2)}$$

$$= \frac{\alpha\beta}{(\beta + \gamma)} \frac{(2\beta + \gamma)}{(2\beta + \gamma)(2\beta - \gamma)} = \frac{\alpha\beta}{(\beta + \gamma)(2\beta - \gamma)} = \frac{\alpha\beta}{2\beta^2 + \beta\gamma - \gamma^2}$$

$$\frac{\partial(y'(p_{x'}^{N}, p_{y'}^{N}))}{\partial \gamma} = \frac{\partial(x'(p_{x'}^{N}, p_{y'}^{N}))}{\partial \gamma} = \alpha\beta \frac{(2\gamma - \beta)}{(\beta + \gamma)^2(2\beta - \gamma)^2}$$

(25) Equation

$$\pi_{y'}(p_{y'}^N, y') = \pi_{x'}(p_{x'}^N, x') = \frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}(2\beta^2-\beta\gamma-\gamma^2)^2$$

$$= \frac{\alpha^2\beta(2\beta+\gamma)^2(\beta-\gamma)^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2} = \frac{\alpha^2\beta(2\beta+\gamma)^2(\beta-\gamma)^2}{(\beta-\gamma)(\beta+\gamma)(2\beta+\gamma)^2(2\beta-\gamma)^2} = \frac{\alpha^2\beta(\beta-\gamma)}{(\beta+\gamma)(2\beta-\gamma)^2}$$

$$\frac{\partial(\pi_{y'}(p_{y'}^N, y'))}{\partial\gamma} = \frac{\partial(\pi_{x'}(p_{x'}^N, x'))}{\partial\gamma} = \alpha^2\beta\frac{-(\beta+\gamma)(2\beta-\gamma)^2-(\beta-\gamma)(-2(2\beta-\gamma)(\beta+\gamma)+(2\beta-\gamma)^2)}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$= \alpha^2\beta\frac{-(\beta+\gamma)(2\beta-\gamma)-(\beta-\gamma)(-2(\beta+\gamma)+(2\beta-\gamma))}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta\frac{-(\beta+\gamma)(2\beta-\gamma)-(\beta-\gamma)(-2\beta-2\gamma+2\beta-\gamma)}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta\frac{-(\beta+\gamma)(2\beta-\gamma)-(\beta-\gamma)(-3\gamma)}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta\frac{(-\beta-\gamma)(2\beta-\gamma)+3\gamma(\beta-\gamma)}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta\frac{(-2\beta^2+\beta\gamma-2\beta\gamma+\gamma^2)+3\beta\gamma-3\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta\frac{-2\beta^2+2\beta\gamma-2\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= -2\alpha^2\beta\frac{\overbrace{\beta^2-\beta\gamma}^{>0}+\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3} < 0$$

(26) Equation n/a

$$U = \frac{2\alpha^2\beta^2(\beta^2-\gamma^2)(2\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)}{(\beta^2-\gamma^2)^2(4\beta_1\beta_2-\gamma^2)^2}$$

$$-\frac{\alpha^2\beta^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\beta(2\beta^2-\gamma^2-\beta\gamma)^2+\gamma(2\beta^2-\gamma^2-\beta\gamma)^2\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\frac{2\beta^2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)}{(4\beta^2-\gamma^2)^2} - \frac{(\beta+\gamma)\beta^2(2\beta^2-\gamma^2-\beta\gamma)^2}{(4\beta^2-\gamma^2)^2})\right]$$

$$= \frac{\alpha^2\beta^2}{(\beta^2-\gamma^2)^2}\left[\frac{2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)}{(4\beta^2-\gamma^2)^2} - \frac{(\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)^2}{(4\beta^2-\gamma^2)^2})\right]$$

$$= \alpha^2\beta^2\left[\frac{2(2\beta+\gamma)(\beta-\gamma)(\beta+\gamma)(2\beta+\gamma)(\beta-\gamma)}{(\beta-\gamma)^2(\beta+\gamma)^2(2\beta-\gamma)^2(2\beta+\gamma)^2} - \frac{(\beta+\gamma)(2\beta+\gamma)^2(\beta-\gamma)^2}{(\beta-\gamma)^2(\beta+\gamma)^2(2\beta-\gamma)^2(2\beta+\gamma)^2}\right]$$

$$= \alpha^2\beta^2\left[\frac{2}{(\beta+\gamma)(2\beta-\gamma)^2} - \frac{1}{(\beta+\gamma)(2\beta-\gamma)^2}\right]$$

$$\Leftrightarrow$$

$$U = \frac{\alpha^2\beta^2}{(\beta+\gamma)(2\beta-\gamma)^2}$$

$$\frac{\partial U}{\partial \gamma} = \alpha^2\beta^2 \frac{-((2\beta-\gamma)^2 - 2(2\beta-\gamma)(\beta+\gamma))}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$= \alpha^2\beta^2 \frac{-(2\beta-\gamma-2\beta-2\gamma)}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta^2 \frac{3\gamma}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

(27) Equation

$$TS = U + 2\pi$$

$$\frac{\partial TS}{\partial \gamma} = \frac{\partial U}{\partial \gamma} + 2\frac{\partial \pi}{\partial \gamma}$$

$$= \frac{3\alpha^2\beta^2\gamma}{(\beta+\gamma)^2(2\beta-\gamma)^3} - \frac{4\alpha^2\beta(\beta^2-\beta\gamma+\gamma^2)}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \frac{3\alpha^2\beta^2\gamma}{(\beta+\gamma)^2(2\beta-\gamma)^3} - \frac{4\alpha^2\beta^3 - 4\alpha^2\beta^2\gamma + 4\alpha^2\beta\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= \alpha^2\beta\frac{-4\beta^2+7\beta\gamma-4\gamma^2}{(\beta+\gamma)^2(2\beta-\gamma)^3} // no \quad real \quad roots$$

(28) Equation n/a

(29) Equation

$$\frac{\partial(p_{x'}^N - p_x^N)}{\partial\gamma} = \frac{\partial p_{x'}^N}{\partial\gamma}$$
$$= \alpha \frac{-(\beta+2\gamma)(4\beta^2-\gamma^2)+2\gamma(2\beta^2-\beta\gamma-\gamma^2)}{(4\beta^2-\gamma^2)^2}$$
$$= \alpha \frac{-(\beta+2\gamma)(4\beta^2-\gamma^2)+(4\beta^2\gamma-2\beta\gamma^2-2\gamma^3)}{(2\beta+\gamma)^2(2\beta-\gamma)^2}$$
$$= \alpha \frac{-4\beta^3+\beta\gamma^2-8\beta^2\gamma+2\gamma^3+4\beta^2\gamma-2\beta\gamma^2-2\gamma^3}{(2\beta+\gamma)^2(2\beta-\gamma)^2}$$
$$= \alpha \frac{-4\beta^3-4\beta^2\gamma-\beta\gamma^2}{(2\beta+\gamma)^2(2\beta-\gamma)^2}$$
$$= \alpha\beta \frac{-4\beta^2-4\beta\gamma-\gamma^2}{(2\beta+\gamma)^2(2\beta-\gamma)^2}$$
$$= -\alpha\beta \frac{4\beta^2+4\beta\gamma+\gamma^2}{(2\beta+\gamma)^2(2\beta-\gamma)^2}$$
$$= -\alpha\beta \frac{(2\beta+\gamma)^2}{(2\beta+\gamma)^2(2\beta-\gamma)^2}$$
$$= \frac{-\alpha\beta}{(2\beta-\gamma)^2}$$

(30) Equation

Normal duopoly

$$p_{y'}^N = \frac{2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2}{4\beta_1\beta_2-\gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2-\gamma^2}c_y + \frac{\beta_2\gamma}{4\beta_1\beta_2-\gamma^2}c_x$$
$$p_{x'}^N = \frac{2\alpha_1\beta_1\beta_2-\alpha_2\beta_1\gamma-\alpha_1\gamma^2}{4\beta_1\beta_2-\gamma^2} + \frac{2\beta_1\beta_2}{4\beta_1\beta_2-\gamma^2}c_x + \frac{\beta_1\gamma}{4\beta_1\beta_2-\gamma^2}c_y$$

Mixed duopoly

$$p_y^N = \frac{\beta_1\alpha_2-\alpha_1\gamma}{2\beta_1} + \frac{c_y}{2} + \frac{\gamma}{2\beta_1}c_x$$
$$p_x^N = c_x$$

$$p_{y'}^{N}-p_{y}^{N}=\begin{bmatrix} \dfrac{2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2}{4\beta_1\beta_2-\gamma^2}-\dfrac{\beta_1\alpha_2-\alpha_1\gamma}{2\beta_1} \\ \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2-\gamma^2}c_y-\dfrac{c_y}{2} \\ \dfrac{\beta_2\gamma}{4\beta_1\beta_2-\gamma^2}c_x-\dfrac{\gamma}{2\beta_1}c_x \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2}{4\beta_1\beta_2-\gamma^2}-\dfrac{\beta_1\alpha_2-\alpha_1\gamma}{2\beta_1} \\ \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2-\gamma^2}c_y-\dfrac{c_y}{2} \\ \dfrac{\beta_2\gamma}{4\beta_1\beta_2-\gamma^2}c_x-\dfrac{\gamma}{2\beta_1}c_x \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2}{4\beta_1\beta_2-\gamma^2}-\dfrac{2\alpha_2\beta_1\beta_2-2\alpha_1\beta_2\gamma-\frac{1}{2}\alpha_2\gamma^2+\frac{1}{2\beta_1}\alpha_1\gamma^3}{4\beta_1\beta_2-\gamma^2} \\ \dfrac{2\beta_1\beta_2}{4\beta_1\beta_2-\gamma^2}c_y-\dfrac{2\beta_1\beta_2-\frac{1}{2}\gamma^2}{4\beta_1\beta_2-\gamma^2}c_y \\ \dfrac{\beta_2\gamma}{4\beta_1\beta_2-\gamma^2}c_x-\dfrac{2\beta_2\gamma-\frac{1}{2\beta_1}\gamma^3}{4\beta_1\beta_2-\gamma^2}c_x \end{bmatrix}$$

$$=\frac{\gamma}{\underbrace{4\beta_1\beta_2-\gamma^2}_{>0\,//\,\beta_1\beta_2-\gamma^2>0}}\begin{bmatrix} \alpha_1\beta_2-\frac{1}{2}\alpha_2\gamma-\frac{1}{2\beta_1}\alpha_1\gamma^2 \\ +\frac{1}{2}\gamma c_y+\underbrace{(\tfrac{1}{2\beta_1}\gamma^2-\beta_2)}_{<0\,//\,\beta_1\beta_2-\gamma^2>0}c_x \end{bmatrix}$$

$$p_{y'}^{N}-p_{y}^{N}=\frac{\gamma}{4\beta^2-\gamma^2}\left[\alpha\beta-\tfrac{1}{2}\alpha\gamma-\tfrac{1}{2\beta}\alpha\gamma^2\right]$$

$$=\frac{\alpha\gamma}{4\beta^2-\gamma^2}\left[\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2\right]//*\,\beta$$

$$=\frac{\alpha\gamma}{\beta(4\beta^2-\gamma^2)}\left[-\tfrac{1}{2}\gamma^2-\tfrac{1}{2}\gamma\beta+\beta^2\right]$$

$$find \quad roots$$

$$\gamma = \frac{\frac{1}{2}\beta \pm \sqrt{\frac{1}{4}\beta^2 + \frac{1}{2}\beta^2}}{-1}$$
$$= \frac{\frac{1}{2}\beta \pm \frac{3}{2}\beta}{-1}$$
$$\Leftrightarrow$$
$$\gamma = \beta(\vee \gamma = -2\beta)$$

$$F.O.C$$
$$\frac{d(-\frac{1}{2}\gamma^2 - \frac{1}{2}\gamma\beta + \beta^2)}{d\gamma} = -\gamma - \frac{1}{2}\beta$$
$$\frac{d^2(-\frac{1}{2}\gamma^2 - \frac{1}{2}\gamma\beta + \beta^2)}{d\gamma^2} = -1$$
$$\Rightarrow$$
$$p_{y'}^N - p_y^N = \frac{\alpha\gamma}{\beta(4\beta^2 - \gamma^2)}\left[-\frac{1}{2}\gamma^2 - \frac{1}{2}\gamma\beta + \beta^2\right]$$
$$= \frac{\alpha\gamma}{\beta(4\beta^2 - \gamma^2)}\left[(\beta - \gamma)(\beta + \frac{1}{2}\gamma)\right]$$

(31) Equation

$$\frac{\partial(p_{y'}^N - p_y^N)}{\partial\gamma}$$
$$= \frac{\alpha\gamma}{\beta(4\beta^2 - \gamma^2)}\left[-(\beta + \frac{1}{2}\gamma) + \frac{1}{2}(\beta - \gamma)\right] + \alpha\frac{(4\beta^2 - \gamma^2) + 2\gamma^2}{\beta(4\beta^2 - \gamma^2)^2}\left[(\beta - \gamma)(\beta + \frac{1}{2}\gamma)\right]$$
$$= \frac{\alpha}{\beta(4\beta^2 - \gamma^2)^2}\left[-(\frac{1}{2}\beta + \gamma)\gamma(4\beta^2 - \gamma^2)\right] + \frac{\alpha}{\beta(4\beta^2 - \gamma^2)^2}\left[(4\beta^2 + \gamma^2)(\beta - \gamma)(\beta + \frac{1}{2}\gamma)\right]$$
$$= \frac{\alpha}{\beta(4\beta^2 - \gamma^2)^2}\left[(4\beta^2 + \gamma^2)(\beta - \gamma)(\beta + \frac{1}{2}\gamma) - \gamma(\frac{1}{2}\beta + \gamma)(4\beta^2 - \gamma^2)\right]$$
$$= \frac{\alpha}{\beta(4\beta^2 - \gamma^2)^2}\left[(4\beta^2 + \gamma^2)(\beta^2 - \frac{1}{2}\beta\gamma - \frac{1}{2}\gamma^2) - (\frac{1}{2}\beta\gamma + \gamma^2)(4\beta^2 - \gamma^2)\right]$$
$$= \frac{\alpha}{\beta(4\beta^2 - \gamma^2)^2}\begin{bmatrix} 4\beta^4 - 2\beta^3\gamma - 2\beta^2\gamma^2 + \beta^2\gamma^2 - \frac{1}{2}\beta\gamma^3 - \frac{1}{2}\gamma^4 \\ -2\beta^3\gamma + \frac{1}{2}\beta\gamma^3 - 4\beta^2\gamma^2 + \gamma^4 \end{bmatrix}$$
$$= \frac{\alpha}{\beta(4\beta^2 - \gamma^2)^2}\left[4\beta^4 - 4\beta^3\gamma - 5\beta^2\gamma^2 + \frac{1}{2}\gamma^4\right]$$
$$= \frac{\alpha}{\beta(2\beta - \gamma)^2(2\beta + \gamma)^2}\left[(2\beta + \gamma)(2\beta^3 - 3\beta^2\gamma - \beta\gamma^2 + \frac{1}{2}\gamma^3)\right]$$

$$
\begin{aligned}
&= \frac{\alpha}{\beta(2\beta-\gamma)^2(2\beta+\gamma)}\left[(2\beta^3 - 3\beta^2\gamma - \beta\gamma^2 + \frac{1}{2}\gamma^3)\right] \\
&= \frac{\alpha}{\beta(2\beta-\gamma)^2(2\beta+\gamma)}\left[(2\beta+\gamma)(\beta^2 - 2\beta\gamma + \frac{1}{2}\gamma^2\right] \\
&= \frac{\alpha}{\beta(2\beta-\gamma)^2}\left[\beta^2 - 2\beta\gamma + \frac{1}{2}\gamma^2\right] \\
&= \frac{\alpha}{\beta(2\beta-\gamma)^2}(\beta - \underbrace{\frac{\sqrt{2}+1}{\sqrt{2}}}_{\approx 1,707}\gamma)(\beta - \underbrace{\frac{\sqrt{2}-1}{\sqrt{2}}}_{\approx 0,293}\gamma)
\end{aligned}
$$

(32) Equation (normal duopoly)

$$
y'(p_{x'}^N, p_{y'}^N) = \frac{\beta_1}{(\beta_1\beta_2 - \gamma^2)} \frac{2\alpha_2\beta_1\beta_2 - \alpha_2\gamma^2 - \alpha_1\beta_2\gamma - \beta_2\gamma c_x{}^2 + \gamma^2 c_y - 2\beta_1\beta_2 c_y}{(4\beta_1\beta_2 - \gamma^2)}
$$

$$
= \frac{\beta}{(\beta^2 - \gamma^2)} \frac{2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma}{(4\beta^2 - \gamma^2)}
$$

Similarly

$$
x'(p_{x'}^N, p_{y'}^N) = \frac{\beta}{(\beta^2 - \gamma^2)} \frac{2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma}{(4\beta^2 - \gamma^2)}
$$

(33) Equation (mixed duopoly)

$$
y(p_x^N, p_y^N) = \frac{\frac{1}{2}(\beta_1\alpha_2 - \alpha_1\gamma)}{\beta_1\beta_2 - \gamma^2} - \frac{\frac{1}{2}\beta_1}{\beta_1\beta_2 - \gamma^2} c_y + \frac{\frac{1}{2}\gamma}{\beta_1\beta_2 - \gamma^2} c_x = \frac{\frac{1}{2}(\beta\alpha - \alpha\gamma)}{\beta^2 - \gamma^2}
$$

Similarly

$$
x(p_x^N, p_y^N) = \frac{\beta_2\alpha_1 - \frac{1}{2}\alpha_2\gamma - \frac{1}{2\beta_1}\alpha_1\gamma^2}{\beta_1\beta_2 - \gamma^2} - \frac{(\beta_2 - \frac{\gamma^2}{2\beta_1})}{\beta_1\beta_2 - \gamma^2} c_x + \frac{\frac{\gamma}{2}}{\beta_1\beta_2 - \gamma^2} c_y = \frac{\beta\alpha - \frac{1}{2}\alpha\gamma - \frac{1}{2\beta}\alpha\gamma^2}{\beta^2 - \gamma^2}
$$

(34) Equation

$$x(p_x^N, p_y^N) - x'(p_{x'}^N, p_{y'}^N) = \frac{\beta\alpha - \frac{1}{2}\alpha\gamma - \frac{1}{2\beta}\alpha\gamma^2}{\beta^2 - \gamma^2} - \frac{\beta}{(\beta^2-\gamma^2)}\frac{2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma}{(4\beta^2-\gamma^2)}$$

$$= \frac{1}{(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\begin{bmatrix} (\beta\alpha - \frac{1}{2}\alpha\gamma - \frac{1}{2\beta}\alpha\gamma^2)(4\beta^2-\gamma^2) \\ -\beta(2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma) \end{bmatrix}$$

$$= \frac{1}{(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\begin{bmatrix} 4\alpha\beta^3 - \alpha\beta\gamma^2 - 2\alpha\gamma\beta^2 + \frac{1}{2}\alpha\gamma^3 \\ -2\alpha\beta\gamma^2 + \frac{1}{2\beta}\alpha\gamma^4 \\ -2\alpha\beta^3 + \alpha\beta\gamma^2 + \alpha\beta^2\gamma) \end{bmatrix}$$

$$= \frac{\alpha}{(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\begin{bmatrix} 4\beta^3 - \beta\gamma^2 - 2\beta^2\gamma + \frac{1}{2}\gamma^3 \\ -2\beta\gamma^2 + \frac{1}{2\beta}\gamma^4 \\ -2\beta^3 + \beta\gamma^2 + \beta^2\gamma) \end{bmatrix}$$

$$= \frac{\alpha}{(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\left[2\beta^3 - 2\beta\gamma^2 - \beta^2\gamma + \frac{1}{2}\gamma^3 + \frac{1}{2\beta}\gamma^4)\right]$$

$$= \frac{\alpha}{\beta(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\left[2\beta^4 - \beta^3\gamma - 2\beta^2\gamma^2 + \frac{1}{2}\beta\gamma^3 + \frac{1}{2}\gamma^4)\right]$$

$$= \frac{\alpha}{2\beta(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\left[4\beta^4 - 2\beta^3\gamma - 4\beta^2\gamma^2 + \beta\gamma^3 + \gamma^4)\right]$$

$$= \frac{2\alpha}{\beta(4\beta^2-\gamma^2)(\beta^2-\gamma^2)}\left[(\beta + \frac{1}{\sqrt{2}}\gamma)(\beta - \frac{1}{\sqrt{2}}\gamma)(\beta + \frac{1}{2}\gamma)(\beta - \gamma)\right]$$

$$= \frac{\alpha}{\beta(2\beta-\gamma)(2\beta+\gamma)(\beta-\gamma)(\beta+\gamma)}\left[(\beta + \frac{1}{\sqrt{2}}\gamma)(\beta - \frac{1}{\sqrt{2}}\gamma)(2\beta + \gamma)(\beta - \gamma)\right]$$

$$= \frac{\alpha}{\beta(2\beta-\gamma)(\beta+\gamma)}\left[(\beta + \frac{1}{\sqrt{2}}\gamma)(\beta - \frac{1}{\sqrt{2}}\gamma)\right]$$

(35) Equation

$$y(p_x^N, p_y^N) - y'(p_{x'}^N, p_{y'}^N) = \frac{\frac{1}{2}(\beta\alpha - \alpha\gamma)}{\beta^2 - \gamma^2} - \frac{\beta}{(\beta^2 - \gamma^2)} \frac{2\alpha\beta^2 - \alpha\gamma^2 - \alpha\beta\gamma}{(4\beta^2 - \gamma^2)}$$

$$= \frac{\alpha}{2(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)} \left[(\beta - \gamma)(4\beta^2 - \gamma^2) - 2\beta(2\beta^2 - \gamma^2 - \beta\gamma)\right]$$

$$= \frac{\alpha}{2(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)} \left[4\beta^3 - \beta\gamma^2 - 4\beta^2\gamma + \gamma^3 - 4\beta^3 + 2\beta\gamma^2 + 2\beta^2\gamma\right]$$

$$= \frac{\alpha}{2(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)} \left[\beta\gamma^2 - 2\beta^2\gamma + \gamma^3\right]$$

$$= \frac{\alpha\gamma}{2(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)} \left[\beta\gamma - 2\beta^2 + \gamma^2\right]$$

$$= \frac{\alpha\gamma}{(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)} \left[-2\beta^2 + \beta\gamma + \gamma^2\right]$$

$$= \frac{\alpha\gamma}{(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)} \left[-(\beta - \gamma)(\beta + \frac{1}{2}\gamma)\right]$$

$$= -\frac{\alpha\gamma}{(\beta - \gamma)(\beta + \gamma)(4\beta^2 - \gamma^2)} \left[(\beta - \gamma)(\beta + \frac{1}{2}\gamma)\right]$$

$$= -\frac{\alpha\gamma}{(\beta + \gamma)(4\beta^2 - \gamma^2)} \left[(\beta + \frac{1}{2}\gamma)\right]$$

$$= -\frac{\alpha\gamma}{(\beta + \gamma)(2\beta - \gamma)(2\beta + \gamma)} \left[(\beta + \frac{1}{2}\gamma)\right]$$

$$= -\frac{\alpha\gamma}{2(\beta + \gamma)(2\beta - \gamma)(\beta + \frac{1}{2}\gamma)} \left[(\beta + \frac{1}{2}\gamma)\right]$$

$$= -\frac{\alpha\gamma}{2(\beta + \gamma)(2\beta - \gamma)}$$

(36) Equation

$$
\begin{aligned}
& x(p_x^N, p_y^N) + y(p_x^N, p_y^N) - (x'(p_{x'}^N, p_{y'}^N) + y'(p_{x'}^N, p_{y'}^N)) \\
&= x(p_x^N, p_y^N) - x'(p_{x'}^N, p_{y'}^N) + y(p_x^N, p_y^N) - y'(p_{x'}^N, p_{y'}^N) \\
&= \frac{\alpha}{\beta(2\beta-\gamma)(\beta+\gamma)}\left[(\beta+\frac{1}{\sqrt{2}}\gamma)(\beta-\frac{1}{\sqrt{2}}\gamma)\right] - \frac{\alpha\gamma}{2(2\beta-\gamma)(\beta+\gamma)} \\
&= \frac{\alpha}{(2\beta-\gamma)(\beta+\gamma)}\left[\frac{(\beta+\frac{1}{\sqrt{2}}\gamma)(\beta-\frac{1}{\sqrt{2}}\gamma)}{\beta} - \frac{\gamma}{2}\right] \\
&= \frac{\alpha}{(2\beta-\gamma)(\beta+\gamma)}\left[\frac{\beta^2-\frac{1}{2}\gamma^2}{\beta} - \frac{\gamma}{2}\right] \\
&= \frac{\alpha}{(2\beta-\gamma)(\beta+\gamma)}\left[\frac{2\beta^2-\gamma^2}{2\beta} - \frac{\beta\gamma}{2\beta}\right] \\
&= \frac{\alpha}{2\beta(2\beta-\gamma)(\beta+\gamma)}\left[2\beta^2-\beta\gamma-\gamma^2\right] \\
&= \frac{\alpha(2\beta+\gamma)(\beta-\gamma)}{2\beta(2\beta-\gamma)(\beta+\gamma)}
\end{aligned}
$$

(37) Equation

$$\frac{\partial(x(p_x^N, p_y^N) - x'(p_{x'}^N, p_{y'}^N))}{\partial\gamma} = \partial \frac{\alpha(\beta^2 - \frac{1}{2}\gamma^2)}{\beta(2\beta-\gamma)(\beta+\gamma)} / \partial\gamma$$

$$= \partial \frac{\alpha}{\beta} \frac{\beta^2 - \frac{1}{2}\gamma^2}{(2\beta-\gamma)(\beta+\gamma)} / \partial\gamma$$

$$= \partial \frac{\alpha}{\beta} \frac{\beta^2 - \frac{1}{2}\gamma^2}{(2\beta-\gamma)(\beta+\gamma)} / \partial\gamma$$

$$= \frac{\alpha}{\beta} \frac{-\gamma(2\beta-\gamma)(\beta+\gamma) - (\beta^2 - \frac{1}{2}\gamma^2)((2\beta-\gamma)-(\beta+\gamma))}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{\beta} \frac{-\gamma(2\beta-\gamma)(\beta+\gamma) - (\beta^2 - \frac{1}{2}\gamma^2)(\beta-2\gamma)}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{\beta} \frac{(-2\beta\gamma+\gamma^2)(\beta+\gamma) - (\beta^3 - 2\beta^2\gamma - \frac{1}{2}\beta\gamma^2 + \gamma^3)}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{\beta} \frac{-2\beta^2\gamma - 2\beta\gamma^2 + \beta\gamma^2 + \gamma^3 - \beta^3 + 2\beta^2\gamma + \frac{1}{2}\beta\gamma^2 - \gamma^3}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{\beta} \frac{-2\beta^2\gamma - \beta\gamma^2 - \beta^3 + 2\beta^2\gamma + \frac{1}{2}\beta\gamma^2}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{1} \frac{-\beta^2 - \frac{1}{2}\gamma^2}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= -\frac{\alpha(\beta^2 + \frac{1}{2}\gamma^2)}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

(38) Equation

$$\frac{\partial(y(p_x^N, p_y^N) - y'(p_{x'}^N, p_{y'}^N))}{\partial\gamma} = \frac{\alpha}{2}\frac{(2\beta-\gamma)(\beta+\gamma)-\gamma((2\beta-\gamma)-(\beta+\gamma))}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{2}\frac{(2\beta-\gamma)(\beta+\gamma)-\gamma(\beta-2\gamma)}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{2}\frac{2\beta^2+2\beta\gamma-\beta\gamma-\gamma^2-\beta\gamma+2\gamma^2}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha}{2}\frac{2\beta^2+\gamma^2}{(2\beta-\gamma)^2(\beta+\gamma)^2}$$

$$= \frac{\alpha(2\beta^2+\gamma^2)}{2(2\beta-\gamma)^2(\beta+\gamma)^2}$$

(39) Equation

$$\frac{\beta^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}(2\alpha\beta+\alpha\gamma)(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)$$

$$+\frac{\beta^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}(2\alpha\beta+\alpha\gamma)(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)$$

$$-\frac{\beta^2}{2(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\begin{bmatrix}\beta(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)^2 \\ +2\gamma(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma) \\ +\beta(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)^2\end{bmatrix}$$

$$= \frac{2\beta^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}(2\alpha\beta+\alpha\gamma)(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)$$

$$-\frac{\beta^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[(\beta+\gamma)(2\alpha\beta^2-\alpha\gamma^2-\alpha\beta\gamma)^2\right]$$

$$= \frac{\alpha^2\beta^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)-(\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)^2\right]$$

$$= \frac{\alpha^2\beta^2}{(\beta^2-\gamma^2)^2}\left[\frac{2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)}{(4\beta^2-\gamma^2)^2}-\frac{(\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)^2}{(4\beta^2-\gamma^2)^2})\right]$$

(40) Equation

$$\frac{\alpha}{\beta^2-\gamma^2}\left[\beta\alpha-\tfrac{1}{2}\alpha\gamma-\tfrac{1}{2\beta}\alpha\gamma^2\right]$$

$$+\frac{(\frac{\alpha\beta+\alpha\gamma}{2\beta})}{\beta^2-\gamma^2}\left[\tfrac{1}{2}(\beta\alpha-\alpha\gamma)\right]$$

$$-\frac{1}{2(\beta^2-\gamma^2)^2}\begin{bmatrix}\beta\left(\beta\alpha-\tfrac{1}{2}\alpha\gamma-\tfrac{1}{2\beta}\alpha\gamma^2\right)^2\\+2\gamma\left(\beta\alpha-\tfrac{1}{2}\alpha\gamma-\tfrac{1}{2\beta}\alpha\gamma^2\right)\left(\tfrac{1}{2}(\beta\alpha-\alpha\gamma)\right)\\\beta\left(\tfrac{1}{2}(\beta\alpha-\alpha\gamma)\right)^2\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[(\beta^2-\gamma^2)(\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2)\right]$$

$$+\frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\frac{(\beta^2-\gamma^2)^2}{4\beta}\right]$$

$$-\frac{\alpha^2}{(\beta^2-\gamma^2)^2}\begin{bmatrix}\tfrac{1}{2}\beta\left(\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2\right)^2\\+\tfrac{1}{2}\gamma\left(\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2\right)(\beta-\gamma)\\\tfrac{1}{8}\beta\left((\beta-\gamma)\right)^2\end{bmatrix}$$

$$=$$

$$\frac{\alpha^2}{(\beta^2-\gamma^2)^2}\begin{bmatrix}(\beta^2-\gamma^2)(\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2)\\+\frac{(\beta^2-\gamma^2)^2}{4\beta}\\-\tfrac{1}{2}\beta\left(\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2\right)^2\\-\tfrac{1}{2}\gamma\left(\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2\beta}\gamma^2\right)(\beta-\gamma)\\-\tfrac{1}{8}\beta(\beta-\gamma)^2\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta^2-\gamma^2)^2}\begin{bmatrix}\frac{1}{2\beta}(\beta^2-\gamma^2)(2\beta^2-\beta\gamma-\gamma^2)\\+\frac{(\beta^2-\gamma^2)^2}{4\beta}\\-\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2\\-\frac{\gamma}{4\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)(\beta-\gamma)\\-\frac{\beta(\beta-\gamma)^2(\beta+\gamma)^2}{8(\beta+\gamma)^2}\end{bmatrix}$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\begin{array}{l}(2\beta^2-\beta\gamma-\gamma^2)(\frac{1}{2\beta}(\beta^2-\gamma^2)-\frac{\gamma}{4\beta}(\beta-\gamma)) \\ -\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta^2-\gamma^2)^2(\frac{1}{4\beta}-\frac{\beta}{8(\beta+\gamma)^2})\end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\begin{array}{l}(2\beta^2-\beta\gamma-\gamma^2)(\frac{2\beta^2-2\gamma^2-\beta\gamma+\gamma^2}{4\beta}) \\ -\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta^2-\gamma^2)^2(\frac{2(\beta+\gamma)^2-\beta^2}{8\beta(\beta+\gamma)^2})\end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\begin{array}{l}\frac{(2\beta^2-\gamma^2-\beta\gamma)^2}{4\beta} \\ -\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta-\gamma)^2(\beta+\gamma)^2(\frac{2(\beta+\gamma)^2-\beta^2}{8\beta(\beta+\gamma)^2})\end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\begin{array}{l}+\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta-\gamma)^2(\frac{2(\beta+\gamma)^2-\beta^2}{8\beta})\end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2}\left[\begin{array}{l}+\frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(\beta-\gamma)^2(\frac{\beta^2+4\beta\gamma+2\gamma^2}{8\beta})\end{array}\right]$$

(41) Equation

$$U_{mixed} - U_{normal}$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2} \left[\begin{array}{l} +\frac{1}{8\beta}\left(2\beta^2 - \beta\gamma - \gamma^2\right)^2 \\ +(\beta-\gamma)^2 (\frac{\beta^2 + 4\beta\gamma + 2\gamma^2}{8\beta}) \end{array} \right]$$

$$- \frac{\alpha^2}{(\beta^2-\gamma^2)^2} \left[\frac{2\beta^2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)}{(4\beta^2-\gamma^2)^2} - \frac{(\beta+\gamma)\beta^2(2\beta^2-\gamma^2-\beta\gamma)^2}{(4\beta^2-\gamma^2)^2}) \right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2} \left[\begin{array}{l} \frac{(4\beta^2-\gamma^2)^2 \frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2}{(4\beta^2-\gamma^2)^2} + \frac{\beta^2(\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)^2}{(4\beta^2-\gamma^2)^2} \\ + \frac{(4\beta^2-\gamma^2)^2(\beta-\gamma)^2(\frac{\beta^2+4\beta\gamma+2\gamma^2}{8\beta})}{(4\beta^2-\gamma^2)^2} - \frac{2\beta^2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma)}{(4\beta^2-\gamma^2)^2} \end{array} \right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2} \left[\begin{array}{l} (4\beta^2-\gamma^2)^2 \frac{1}{8\beta}\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 + \beta^2(\beta+\gamma)(2\beta^2-\gamma^2-\beta\gamma)^2 \\ +(4\beta^2-\gamma^2)^2(\beta-\gamma)^2(\frac{\beta^2+4\beta\gamma+2\gamma^2}{8\beta}) - 2\beta^2(2\beta+\gamma)(\beta^2-\gamma^2)(2\beta^2-\gamma^2-\beta\gamma) \end{array} \right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2} \left[\begin{array}{l} ((16\beta^4 - 8\gamma^2\beta^2 + \gamma^4)\frac{1}{8\beta} + \beta^3 + \beta^2\gamma))\left(2\beta^2-\beta\gamma-\gamma^2\right)^2 \\ +(16\beta^4 - 8\gamma^2\beta^2 + \gamma^4)\frac{1}{8\beta}(\beta^2 - 2\gamma\beta + \gamma^2)(\beta^2 + 4\beta\gamma + 2\gamma^2) \\ -(4\beta^3 + 2\beta^2\gamma)(2\beta^4 - \beta^2\gamma^2 - \beta^3\gamma - 2\beta^2\gamma^2 + \gamma^4 + \beta\gamma^3) \end{array} \right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2} \left[\begin{array}{l} (3\beta^3 + \beta^2\gamma - \gamma^2\beta + \frac{\gamma^4}{8\beta})\left(4\beta^4 - 4\beta^3\gamma - 3\beta^2\gamma^2 + 2\beta\gamma^3 + \gamma^4\right) \\ (2\beta^3 - \beta\gamma^2 + \frac{\gamma^4}{8\beta})(\beta^4 + 2\beta^3\gamma - 5\gamma^2\beta^2 + 2\gamma^4) \\ -(4\beta^3 + 2\beta^2\gamma)(2\beta^4 - 3\beta^2\gamma^2 - \beta^3\gamma + \gamma^4 + \beta\gamma^3) \end{array} \right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2} \left[\begin{array}{l} +12\beta^7 - 12\beta^6\gamma - 9\beta^5\gamma^2 + 6\beta^4\gamma^3 + 3\beta^3\gamma^4 \\ +4\beta^6\gamma - 4\beta^5\gamma^2 - 3\beta^4\gamma^3 + 2\beta^3\gamma^4 + \beta^2\gamma^5 \\ -4\beta^5\gamma^2 + 4\beta^4\gamma^3 + 3\beta^3\gamma^4 - 2\beta^2\gamma^5 - \beta\gamma^6 \\ +\frac{1}{2}\beta^3\gamma^4 - \frac{1}{2}\beta^2\gamma^5 - \frac{3}{8}\beta\gamma^6 + \frac{1}{4}\gamma^7 + \frac{1}{8\beta}\gamma^8 \\ (2\beta^3 - \beta\gamma^2 + \frac{\gamma^4}{8\beta})(\beta^4 + 2\beta^3\gamma - 5\gamma^2\beta^2 + 2\gamma^4) \\ -8\beta^7 + 12\beta^5\gamma^2 + 4\beta^6\gamma - 4\beta^3\gamma^4 - 4\beta^4\gamma^3 \\ -4\beta^6\gamma + 6\beta^4\gamma^3 + 2\beta^5\gamma^2 - 2\beta^2\gamma^5 - 2\beta^3\gamma^4) \end{array} \right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} +12\beta^7 -12\beta^6\gamma -9\beta^5\gamma^2 +6\beta^4\gamma^3 +3\beta^3\gamma^4 \\ +4\beta^6\gamma -4\beta^5\gamma^2 -3\beta^4\gamma^3 +2\beta^3\gamma^4 +\beta^2\gamma^5 \\ -4\beta^5\gamma^2 +4\beta^4\gamma^3 +3\beta^3\gamma^4 -2\beta^2\gamma^5 -\beta\gamma^6 \\ +\frac{1}{2}\beta^3\gamma^4 -\frac{1}{2}\beta^2\gamma^5 -\frac{3}{8}\beta\gamma^6 +\frac{1}{4}\gamma^7 +\frac{1}{8\beta}\gamma^8 \\ 2\beta^7 +4\beta^6\gamma -10\beta^5\gamma^2 +4\beta^3\gamma^4 \\ -\beta^5\gamma^2 -2\beta^4\gamma^3 +5\beta^3\gamma^4 -2\beta\gamma^6 \\ +\frac{1}{8}\beta^3\gamma^4 +\frac{1}{4}\beta^2\gamma^5 -\frac{5}{8}\beta\gamma^6 +\frac{1}{4\beta}\gamma^8 \\ -8\beta^7 +12\beta^5\gamma^2 +4\beta^6\gamma -4\beta^3\gamma^4 -4\beta^4\gamma^3 \\ -4\beta^6\gamma +6\beta^4\gamma^3 +2\beta^5\gamma^2 -2\beta^2\gamma^5 -2\beta^3\gamma^4 \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} +6\beta^7 -4\beta^6\gamma -14\beta^5\gamma^2 +7\beta^4\gamma^3 +11\beta^3\gamma^4 \\ -3\beta^2\gamma^5 -4\beta\gamma^6 \\ +\frac{1}{4}\gamma^7 +\frac{3}{8\beta}\gamma^8 \\ +\frac{5}{8}\beta^3\gamma^4 -\frac{1}{4}\beta^2\gamma^5 \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} +6\beta^7 -4\beta^6\gamma -14\beta^5\gamma^2 +7\beta^4\gamma^3 +11\beta^3\gamma^4 \\ -3\beta^2\gamma^5 -4\beta\gamma^6 \\ +\frac{1}{4}\gamma^7 +\frac{3}{8\beta}\gamma^8 \\ +\frac{5}{8}\beta^3\gamma^4 -\frac{1}{4}\beta^2\gamma^5 \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^7 - 14\beta^5\gamma^2 \\ 7\beta^4\gamma^3 - 3\beta^2\gamma^5 \\ -4\beta\gamma^6 + \frac{3}{8\beta}\gamma^8 \\ -\frac{1}{4}\beta^2\gamma^5 + \frac{1}{4}\gamma^7 \\ +\frac{93}{8}\beta^3\gamma^4 - 4\beta^6\gamma \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^7 - 14\beta^5\gamma^2 \\ 7\beta^4\gamma^3 - 3\beta^2\gamma^5 \\ -4\beta\gamma^6 + \frac{3}{8\beta}\gamma^8 \\ -\frac{1}{4}\beta^2\gamma^5 + \frac{1}{4}\gamma^7 \\ +\frac{93}{8}\beta^3\gamma^4 - 4\beta^6\gamma \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^7 - 6\beta^5\gamma^2 \\ 3\beta^4\gamma^3 - 3\beta^2\gamma^5 \\ -\frac{3}{8}\beta\gamma^6 + \frac{3}{8\beta}\gamma^8 \\ -\frac{1}{4}\beta^2\gamma^5 + \frac{1}{4}\gamma^7 \\ 4\beta^4\gamma^3 - 8\beta^5\gamma^2 - \frac{29}{8}\beta\gamma^6 \\ +\frac{93}{8}\beta^3\gamma^4 - 4\beta^6\gamma \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\begin{bmatrix} (\beta^2-\gamma^2)(6\beta^5+3\beta^2\gamma^3-\frac{3}{8\beta}\gamma^6-\frac{1}{4}\gamma^5) \\ -8\beta^5\gamma^2+8\beta^3\gamma^4 \\ -\frac{29}{8}\beta\gamma^6+\frac{29}{8}\beta^3\gamma^4 \\ -4\beta^6\gamma+4\beta^4\gamma^3 \end{bmatrix}$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)^2(4\beta^2-\gamma^2)^2}\left[(\beta^2-\gamma^2)(-4\beta^4\gamma+\frac{29}{8}\beta\gamma^4-8\beta^3\gamma^2+6\beta^5+3\beta^2\gamma^3-\frac{3}{8\beta}\gamma^6-\frac{1}{4}\gamma^5)\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left[-4\beta^4\gamma+\frac{29}{8}\beta\gamma^4-8\beta^3\gamma^2+6\beta^5+3\beta^2\gamma^3-\frac{3}{8\beta}\gamma^6-\frac{1}{4}\gamma^5\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left[6\beta^5+3\beta^2\gamma^3+\frac{29}{8}\beta\gamma^4-4\beta^4\gamma-8\beta^3\gamma^2-\frac{3}{8\beta}\gamma^6-\frac{1}{4}\gamma^5\right]$$

$$= \frac{\alpha^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\begin{bmatrix} 4\beta^5-4\beta^4\gamma \\ -3\beta^3\gamma^2+3\beta^2\gamma^3 \\ 2\beta^5-5\beta^3\gamma^2 \\ +\frac{29}{8}\beta\gamma^4-\frac{3}{8\beta}\gamma^6-\frac{1}{4}\gamma^5 \end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\begin{bmatrix}(\beta-\gamma)(4\beta^4-3\beta^2\gamma^2+\frac{1}{4}\gamma^4)\\+(\beta^2-\gamma^2)(2\beta^3-3\beta\gamma^2+\frac{3}{8\beta}\gamma^4)\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}4\beta^4-3\beta^2\gamma^2+\frac{1}{4}\gamma^4\\+(\beta+\gamma)(2\beta^3-3\beta\gamma^2+\frac{3}{8\beta}\gamma^4)\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}4\beta^4-3\beta^2\gamma^2+\frac{1}{4}\gamma^4\\2\beta^4-3\beta^2\gamma^2+\frac{3}{8}\gamma^4\\+2\beta^3\gamma-3\beta\gamma^3+\frac{3}{8\beta}\gamma^5\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^4-6\beta^2\gamma^2+\frac{5}{8}\gamma^4\\+2\beta^3\gamma-3\beta\gamma^3+\frac{3}{8\beta}\gamma^5\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^4-6\beta^2\gamma^2+\frac{5}{8}\gamma^4\\+\frac{\gamma}{\beta}(2\beta^4-3\beta^2\gamma^2+\frac{3}{8}\gamma^4)\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^4-6\beta^2\gamma^2+\frac{5}{8}\gamma^4\\+\frac{\gamma}{\beta}((\beta^2-\gamma^2)^2+\beta^4-\beta^2\gamma^2-\frac{5}{8}\gamma^4)\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^4-6\beta^2\gamma^2\\+\frac{5}{8}\gamma^4-\frac{5}{8\beta}\gamma^5\\+\frac{\gamma}{\beta}((\beta^2-\gamma^2)^2+\beta^4-\beta^2\gamma^2)\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6(\beta^4-\beta^2\gamma^2)\\+\frac{5}{8\beta}(\beta\gamma^4-\gamma^5)\\+\frac{\gamma}{\beta}((\beta^2-\gamma^2)^2+\beta^4-\beta^2\gamma^2)\end{bmatrix}$$

$$= \frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^2(\beta^2-\gamma^2) \\ +\dfrac{5\gamma^4}{8\beta}(\beta-\gamma) \\ +\dfrac{\gamma}{\beta}((\beta^2-\gamma^2)^2+\beta^2(\beta^2-\gamma^2)) \end{array}\right]$$

$$= \frac{\alpha^2}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^2(\beta-\gamma)(\beta+\gamma) \\ +\dfrac{5\gamma^4}{8\beta}(\beta-\gamma) \\ +\dfrac{\gamma}{\beta}((\beta-\gamma)^2(\beta+\gamma)^2+\beta^2(\beta-\gamma)(\beta+\gamma)) \end{array}\right]$$

$$= \frac{\alpha^2(\beta-\gamma)}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^2(\beta+\gamma) \\ +\dfrac{5\gamma^4}{8\beta} \\ +\dfrac{\gamma}{\beta}((\beta-\gamma)(\beta+\gamma)^2+\beta^2(\beta+\gamma)) \end{array}\right]$$

$$= \frac{\alpha^2(\beta-\gamma)}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} 6\beta^2(\beta+\gamma) \\ +\dfrac{5\gamma^4}{8\beta} \\ +\dfrac{\gamma(\beta+\gamma)}{\beta}((\beta-\gamma)(\beta+\gamma)+\beta^2) \end{array}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} \dfrac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\dfrac{5\gamma^3}{8} \\ +(\beta+\gamma)(\beta^2-\gamma^2+\beta^2) \end{array}\right]$$

(42) Equation

$$\frac{\partial(U_{mixed}-U_{normal})}{\partial\gamma}=\frac{\partial(\frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2})}{\partial\gamma}\begin{bmatrix}6\beta^3(\beta+\gamma)\\+\frac{5\gamma^4}{8}\\+\gamma(\beta+\gamma)(\beta^2-\gamma^2+\beta^2)\end{bmatrix}$$

$$+\frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\frac{\partial\begin{bmatrix}6\beta^3(\beta+\gamma)\\+\frac{5\gamma^4}{8}\\+\gamma(\beta+\gamma)(\beta^2-\gamma^2+\beta^2)\end{bmatrix}}{\partial\gamma}$$

$$=\frac{-\alpha^2\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2-\alpha^2\beta(\beta-\gamma)((4\beta^2-\gamma^2)^2+(\beta+\gamma)(-4\gamma(4\beta^2-\gamma^2))}{\beta^2(\beta+\gamma)^2(4\beta^2-\gamma^2)^4}\begin{bmatrix}6\beta^3(\beta+\gamma)\\+\frac{5\gamma^4}{8}\\+\gamma(\beta+\gamma)(\beta^2-\gamma^2+\beta^2)\end{bmatrix}$$

$$+\frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^3\\+\frac{5\gamma^3}{2}\\-2\gamma^2(\beta+\gamma)\\+(2\beta^2-\gamma^2)(\beta+2\gamma)\end{bmatrix}$$

$$=\frac{-\alpha^2\beta(4\beta^2-\gamma^2)\left[(\beta+\gamma)(4\beta^2-\gamma^2)+(\beta-\gamma)((4\beta^2-\gamma^2)-4\gamma(\beta+\gamma))\right]}{\beta^2(\beta+\gamma)^2(4\beta^2-\gamma^2)^4}\begin{bmatrix}6\beta^3(\beta+\gamma)\\+\frac{5\gamma^4}{8}\\+\gamma(\beta+\gamma)(\beta^2-\gamma^2+\beta^2)\end{bmatrix}$$

$$+\frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^3\\+\frac{5\gamma^3}{2}\\-2\gamma^2(\beta+\gamma)\\+(2\beta^2-\gamma^2)(\beta+2\gamma)\end{bmatrix}$$

$$=\frac{-2\alpha^2\left[4\beta^3-\beta\gamma^2-2\beta^2\gamma+2\gamma^3\right]}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\begin{bmatrix}6\beta^3(\beta+\gamma)\\+\frac{5\gamma^4}{8}\\+\gamma(\beta+\gamma)(\beta^2-\gamma^2+\beta^2)\end{bmatrix}$$

$$+\frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix}6\beta^3\\+\frac{5\gamma^3}{2}\\-2\gamma^2(\beta+\gamma)\\+(2\beta^2-\gamma^2)(\beta+2\gamma)\end{bmatrix}$$

$$= \frac{-2\alpha^2\left[4\beta^3-\beta\gamma^2-2\beta^2\gamma+2\gamma^3\right]}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[6\beta^4+8\beta^3\gamma-\frac{3}{8}\gamma^4-\beta\gamma^3+2\beta^2\gamma^2\right]$$

$$+\frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[8\beta^3-3\beta\gamma^2-\frac{3}{2}\gamma^3+4\beta^2\gamma\right]$$

$$=\frac{\alpha^2}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[-2(4\beta^3-\beta\gamma^2-2\beta^2\gamma+2\gamma^3)(6\beta^4+8\beta^3\gamma-\frac{3}{8}\gamma^4-\beta\gamma^3+2\beta^2\gamma^2)\right]$$

$$+\frac{\alpha^2}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[(\beta^2-\gamma^2)(8\beta^3-3\beta\gamma^2-\frac{3}{2}\gamma^3+4\beta^2\gamma)(4\beta^2-\gamma^2)\right]$$

$$=\frac{\alpha^2}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[(-8\beta^3+2\beta\gamma^2+4\beta^2\gamma-4\gamma^3)(6\beta^4+8\beta^3\gamma-\frac{3}{8}\gamma^4-\beta\gamma^3+2\beta^2\gamma^2)\right]$$

$$+\frac{\alpha^2}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[(4\beta^4-\beta^2\gamma^2-4\beta^2\gamma^2+\gamma^4)(8\beta^3-3\beta\gamma^2-\frac{3}{2}\gamma^3+4\beta^2\gamma)\right]$$

$$=\frac{\alpha^2}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\begin{bmatrix}-48\beta^7-64\beta^6\gamma+3\beta^3\gamma^4+8\beta^4\gamma^3-16\beta^5\gamma^2\\ +12\beta^5\gamma^2+16\beta^4\gamma^3-\frac{3}{4}\beta\gamma^6-2\beta^2\gamma^5+4\beta^3\gamma^4\\ +24\beta^6\gamma+32\beta^5\gamma^2-\frac{3}{2}\beta^2\gamma^5-4\beta^3\gamma^4+8\beta^4\gamma^3\\ -24\beta^4\gamma^3-32\beta^3\gamma^4+\frac{3}{2}\gamma^7+4\beta\gamma^6-8\beta^2\gamma^5\end{bmatrix}$$

$$+\frac{\alpha^2}{\beta(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\begin{bmatrix}+32\beta^7-12\beta^5\gamma^2-6\beta^4\gamma^3+16\beta^6\gamma\\ -8\beta^5\gamma^2+3\beta^3\gamma^4+\frac{3}{2}\beta^2\gamma^5-4\beta^4\gamma^3\\ -32\beta^5\gamma^2+12\beta^3\gamma^4+6\beta^2\gamma^5-16\beta^4\gamma^3\\ +8\beta^3\gamma^4-3\beta\gamma^6-\frac{3}{2}\gamma^7+4\beta^2\gamma^5\end{bmatrix}$$

$$=\frac{\alpha^2}{(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[\overbrace{-16\beta^6+\frac{1}{4}\gamma^6}^{<0}-24\beta^5\gamma-6\beta^2\gamma^4-18\beta^3\gamma^3-24\beta^4\gamma^2\right]<0$$

$$\frac{\alpha^2}{(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[\begin{array}{l}(\beta+\frac{1}{2}\gamma)(-16\beta^5-16\beta^4\gamma-16\beta^2\gamma^3+\frac{1}{2}\gamma^5-16\beta^3\gamma^2)\\ +2\beta^3\gamma^3-\frac{1}{2}\beta\gamma^5+4\beta^3\gamma^3+2\beta^2\gamma^4\end{array}\right]$$

$$=\frac{\alpha^2}{(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[(\beta+\frac{1}{2}\gamma)(-16\beta^5-16\beta^4\gamma-16\beta^2\gamma^3+\frac{1}{2}\gamma^5-16\beta^3\gamma^2-\beta\gamma^4+6\beta^2\gamma^3)\right]$$

$$=\frac{\alpha^2}{(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[(\beta+\frac{1}{2}\gamma)^2(-16\beta^4-8\beta^3\gamma-12\beta^2\gamma^2+\gamma^4-4\beta\gamma^3)\right]$$

$$=\frac{\alpha^2}{(\beta+\gamma)^2(4\beta^2-\gamma^2)^3}\left[(\beta+\frac{1}{2}\gamma)^3(-16\beta^3-12\beta\gamma^2+2\gamma^3\right]$$

$$=\frac{\alpha^2}{(\beta+\gamma)^2(2\beta-\gamma)^3(2\beta+\gamma)^3}\left[(\beta+\frac{1}{2}\gamma)^3(-16\beta^3-12\beta\gamma^2+2\gamma^3\right]$$

$$=\frac{\alpha^2}{8(\beta+\gamma)^2(2\beta-\gamma)^3(\beta+\frac{1}{2}\gamma)^3}\left[(\beta+\frac{1}{2}\gamma)^3(-16\beta^3-12\beta\gamma^2+2\gamma^3\right]$$

$$=-\frac{\alpha^2}{4(\beta+\gamma)^2(2\beta-\gamma)^3}\left[8\beta^3+6\beta\gamma^2-\gamma^3\right]$$

(43) Equation

$$\pi_{y'}(p_{y'}^N,y')=$$

$$=\frac{\beta_1}{(\beta_1\beta_2-\gamma^2)(4\beta_1\beta_2-\gamma^2)^2}\left(\begin{array}{l}(2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2)^2\\ +(2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2)(\beta_2\gamma)c_x-\beta_2\gamma c_x{}^2(2\alpha_2\beta_1\beta_2-\alpha_1\beta_2\gamma-\alpha_2\gamma^2)\\ -\beta_2\gamma c_x{}^2c_y(2\beta_1\beta_2-\gamma^2)-c_x^3\beta_2^2\gamma^2-c_y^2(2\beta_1\beta_2-\gamma^2)^2-c_xc_y\beta_2\gamma(2\beta_1\beta_2-\gamma^2)\end{array}\right)$$

$$=\frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left((2\beta^2-\beta\gamma-\gamma^2)^2\right)$$

(44) Equation

$$\pi(p_y,y)=\frac{(\beta_1\alpha_2-\alpha_1\gamma-\beta_1c_y+\gamma c_x)^2}{4\beta_1(\beta_1\beta_2-\gamma^2)}$$

$$=\frac{\alpha^2(\beta-\gamma)^2}{4\beta(\beta^2-\gamma^2)}$$

$$=\frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)}\frac{(\beta-\gamma)^2(4\beta^2-\gamma^2)}{4\beta^2}$$

(45) Equation

$$\pi_{y'}(p_{y'}^{N}, y') - \pi(p_y, y) = \left[\begin{array}{l} \dfrac{\alpha^2 \beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left((2\beta^2-\beta\gamma-\gamma^2)^2\right) \\ -\dfrac{\alpha^2 \beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\dfrac{(\beta-\gamma)^2(4\beta^2-\gamma^2)^2}{4\beta^2} \end{array} \right]$$

$$= \frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left[\left((2\beta^2-\beta\gamma-\gamma^2)^2\right)-\frac{(\beta-\gamma)^2(4\beta^2-\gamma^2)^2}{4\beta^2}\right]$$

$$= \frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left[4\beta^4-4\beta^3\gamma-2\beta^2\gamma^2+2\beta\gamma^3+\gamma^4-\beta^2\gamma^2-\frac{(\beta-\gamma)^2(4\beta^2-\gamma^2)^2}{4\beta^2}\right]$$

$$= \frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left[4\beta^4-\beta^2\gamma^2-4\beta^3\gamma-2\beta^2\gamma^2+2\beta\gamma^3+\gamma^4-\frac{(\beta-\gamma)^2(4\beta^2-\gamma^2)^2}{4\beta^2}\right]$$

$$= \frac{\alpha^2\beta}{(\beta^2-\gamma^2)(4\beta^2-\gamma^2)^2}\left[(4\beta^2-\gamma^2)\beta(\beta-\gamma)-\gamma^2(\beta^2-\gamma^2)-\beta\gamma^2(\beta-\gamma)-\frac{(\beta-\gamma)^2(4\beta^2-\gamma^2)^2}{4\beta^2}\right]$$

$$= \frac{\alpha^2\beta}{(\beta-\gamma)(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} (4\beta^2-\gamma^2)\beta(\beta-\gamma)-\gamma^2(\beta-\gamma)(\beta+\gamma)-\beta\gamma^2(\beta-\gamma) \\ -\dfrac{(\beta-\gamma)^2(4\beta^2-\gamma^2)^2}{4\beta^2} \end{array} \right]$$

$$= \frac{\alpha^2\beta}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[(4\beta^2-\gamma^2)\beta-\gamma^2(\beta+\gamma)-\beta\gamma^2-\frac{(\beta-\gamma)(4\beta^2-\gamma^2)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2\beta}{(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[(4\beta^2-\gamma^2)\beta-2\beta\gamma^2+\gamma^3-\frac{(\beta-\gamma)(4\beta^2-\gamma^2)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2\beta}{(\beta+\gamma)(2\beta+\gamma)^2(2\beta-\gamma)^2}\left[(2\beta+\gamma)(2\beta-\gamma)\beta-\gamma^2(2\beta+\gamma)-\frac{(\beta-\gamma)(2\beta+\gamma)^2(2\beta-\gamma)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2\beta}{(\beta+\gamma)(2\beta+\gamma)(2\beta-\gamma)^2}\left[(2\beta-\gamma)\beta-\gamma^2-\frac{(\beta-\gamma)(2\beta+\gamma)(2\beta-\gamma)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2\beta}{(\beta+\gamma)(2\beta+\gamma)(2\beta-\gamma)^2}\left[(2\beta+\gamma)(\beta-\gamma)-\frac{(\beta-\gamma)(2\beta+\gamma)(2\beta-\gamma)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2\beta}{(\beta+\gamma)(2\beta-\gamma)^2}\left[(\beta-\gamma)-\frac{(\beta-\gamma)(2\beta-\gamma)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2\beta(\beta-\gamma)}{(\beta+\gamma)(2\beta-\gamma)^2}\left[1-\frac{(2\beta-\gamma)^2}{4\beta^2}\right]$$

$$=\frac{\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}\left[4\beta^2-(2\beta-\gamma)^2\right]$$

$$=\frac{\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}\left[4\beta^2-(4\beta^2-4\beta\gamma+\gamma^2)\right]$$

$$=\frac{\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}\left[4\beta\gamma-\gamma^2\right]$$

$$=\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}\left[4\beta-\gamma\right]$$

(46) Equation

$$\frac{\partial(\pi_{y'}(p_{y'}^{N},y')-\pi(p_y,y))}{\partial\gamma}$$

$$=\frac{\partial\left(\frac{\alpha^2\gamma(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}\right)}{\partial\gamma}(4\beta-\gamma)+\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}\frac{\partial[4\beta-\gamma]}{\partial\gamma}$$

$$=\frac{\alpha^2}{4\beta}\frac{(\beta-2\gamma)(\beta+\gamma)(2\beta-\gamma)^2-\gamma(\beta-\gamma)((2\beta-\gamma)^2-2(\beta+\gamma)(2\beta-\gamma))}{(\beta+\gamma)^2(2\beta-\gamma)^4}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}$$

$$=\frac{\alpha^2}{4\beta}\frac{(\beta-2\gamma)(\beta+\gamma)(2\beta-\gamma)-\gamma(\beta-\gamma)((2\beta-\gamma)-2(\beta+\gamma))}{(\beta+\gamma)^2(2\beta-\gamma)^3}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}$$

$$=\frac{\alpha^2}{4\beta}\frac{(\beta-2\gamma)(\beta+\gamma)(2\beta-\gamma)+3\gamma^2(\beta-\gamma)}{(\beta+\gamma)^2(2\beta-\gamma)^3}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}$$

$$=\frac{\alpha^2}{4\beta}\frac{(\beta-2\gamma)(2\beta^2-\beta\gamma+2\beta\gamma-\gamma^2)+3\beta\gamma^2-\gamma^3}{(\beta+\gamma)^2(2\beta-\gamma)^3}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}$$

$$=\frac{\alpha^2}{4\beta}\frac{2\beta^3-\beta^2\gamma+2\beta^2\gamma-\beta\gamma^2-4\beta^2\gamma+2\beta\gamma^2-4\beta\gamma^2+2\gamma^3+3\beta\gamma^2-3\gamma^3}{(\beta+\gamma)^2(2\beta-\gamma)^3}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}$$

$$=\frac{\alpha^2}{4\beta}\frac{2\beta^3-3\beta^2\gamma-\gamma^3}{(\beta+\gamma)^2(2\beta-\gamma)^3}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)}{4\beta(\beta+\gamma)(2\beta-\gamma)^2}$$

$$=\frac{\alpha^2}{4\beta}\frac{2\beta^3-3\beta^2\gamma-\gamma^3}{(\beta+\gamma)^2(2\beta-\gamma)^3}(4\beta-\gamma)-\frac{\gamma\alpha^2(\beta-\gamma)(\beta+\gamma)(2\beta-\gamma)}{4\beta(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$=\frac{\alpha^2}{4\beta(\beta+\gamma)^2(2\beta-\gamma)^3}\left[(2\beta^3-3\beta^2\gamma-\gamma^3)(4\beta-\gamma)-\gamma(\beta-\gamma)(\beta+\gamma)(2\beta-\gamma)\right]$$

$$=\frac{\alpha^2}{4\beta(\beta+\gamma)^2(2\beta-\gamma)^3}\left[(2\beta^3-3\beta^2\gamma-\gamma^3)(4\beta-\gamma)-\gamma(\beta^2-\gamma^2)(2\beta-\gamma)\right]$$

$$=\frac{\alpha^2}{4\beta(\beta+\gamma)^2(2\beta-\gamma)^3}\begin{bmatrix}8\beta^4-12\beta^3\gamma-4\beta\gamma^3-2\beta^3\gamma+3\beta^2\gamma^2+\gamma^4\\+(\gamma^3-\beta^2\gamma)(2\beta-\gamma)\end{bmatrix}$$

$$=\frac{\alpha^2}{4\beta(\beta+\gamma)^2(2\beta-\gamma)^3}\begin{bmatrix}8\beta^4-12\beta^3\gamma-4\beta\gamma^3-2\beta^3\gamma+3\beta^2\gamma^2+\gamma^4\\+2\beta\gamma^3-\gamma^4-2\beta^3\gamma+\beta^2\gamma^2\end{bmatrix}$$

$$=\frac{\alpha^2}{4\beta(\beta+\gamma)^2(2\beta-\gamma)^3}\left[8\beta^4-16\beta^3\gamma-2\beta\gamma^3+4\beta^2\gamma^2\right]$$

$$=\frac{\alpha^2}{2(\beta+\gamma)^2(2\beta-\gamma)^3}\left[4\beta^3-8\beta^2\gamma-\gamma^3+2\beta\gamma^2\right]$$

$$=\frac{\alpha^2}{2(\beta+\gamma)^2(2\beta-\gamma)^3}\left[4(\beta-\overbrace{\frac{1}{1-\frac{1}{\sqrt{5}}}}^{\approx 1,8}\gamma)\overbrace{(\beta^2-\frac{1-\frac{2}{\sqrt{5}}}{1-\frac{1}{\sqrt{5}}}\beta\gamma+\frac{\sqrt{5}-1}{4\sqrt{5}}\gamma^2)}^{>0}\right]$$

(47) Equation

$$
\begin{aligned}
\Delta TS &= TS_{mixed} - TS_{normal} \\
&= (U_{mixed} + \pi_{mixed}) - (2\pi_{normal} + U_{normal}) \\
&= (U_{mixed} - U_{normal}) + (\pi_{mixed} - \pi_{normal}) - \pi_{normal}
\end{aligned}
$$

$$
U_{mixed} - U_{normal}
$$

$$
= \frac{\alpha^2 \gamma(\beta - \gamma)}{\beta(\beta + \gamma)(4\beta^2 - \gamma^2)^2}
\begin{bmatrix}
\dfrac{6\beta^3(\beta + \gamma)}{\gamma} \\
+ \dfrac{5\gamma^3}{8} \\
+ (\beta + \gamma)(2\beta^2 - \gamma^2) \\
\end{bmatrix}
$$

$$
\begin{aligned}
&= \pi_{mixed} - \pi_{normal} \\
&= -(\pi_{normal} - \pi_{mixed}) \\
&= -\frac{\gamma\alpha^2(\beta - \gamma)}{4\beta(\beta + \gamma)(2\beta - \gamma)^2}\left[4\beta - \gamma\right] \\
&= \frac{\gamma\alpha^2(\beta - \gamma)}{\beta(\beta + \gamma)(4\beta^2 - \gamma^2)^2}\left[-\frac{1}{4}(4\beta - \gamma)(2\beta + \gamma)^2\right]
\end{aligned}
$$

$$
\pi_{normal}
$$

$$
\begin{aligned}
&= \frac{\alpha^2\beta}{(\beta^2 - \gamma^2)(4\beta^2 - \gamma^2)^2}\left[(2\beta^2 - \beta\gamma - \gamma^2)^2\right] \\
&= \frac{\alpha^2\beta}{(\beta - \gamma)(\beta + \gamma)(4\beta^2 - \gamma^2)^2}\left[(2\beta^2 - \beta\gamma - \gamma^2)^2\right] \\
&= \frac{\alpha^2\gamma(\beta - \gamma)}{\beta(\beta + \gamma)(4\beta^2 - \gamma^2)^2}\left[\frac{\beta^2(2\beta^2 - \beta\gamma - \gamma^2)^2}{\gamma(\beta - \gamma)^2}\right]
\end{aligned}
$$

$$TS_{mixed} - TS_{normal}$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(2\beta^2-\gamma^2) \end{bmatrix} + \frac{\gamma\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[-\frac{1}{4}(4\beta-\gamma)(2\beta+\gamma)^2\right]$$

$$- \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\frac{\beta^2(2\beta^2-\beta\gamma-\gamma^2)^2}{\gamma(\beta-\gamma)^2}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(2\beta^2-\gamma^2) \\ -\frac{1}{4}(4\beta-\gamma)(2\beta+\gamma)^2 \\ -\frac{\beta^2(2\beta^2-\beta\gamma-\gamma^2)^2}{\gamma(\beta-\gamma)^2} \end{bmatrix}$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\begin{bmatrix} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(2\beta^2-\gamma^2) \\ -\frac{1}{4}(4\beta-\gamma)(2\beta+\gamma)^2 \\ -\frac{\beta^2(2\beta^2-\beta\gamma-\gamma^2)^2}{\gamma(\beta-\gamma)^2} \end{bmatrix}$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(2\beta^2-\gamma^2) \\ -\frac{1}{4}(4\beta-\gamma)(2\beta+\gamma)^2 \\ -\frac{\beta^2(\beta-\gamma)^2(2\beta+\gamma)^2}{\gamma(\beta-\gamma)^2} \end{array}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(2\beta^2-\gamma^2) \\ -\frac{1}{4}(4\beta-\gamma)(2\beta+\gamma)^2 \\ -\frac{\beta^2(2\beta+\gamma)^2}{\gamma} \end{array}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l} \frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +(\beta+\gamma)(2\beta^2-\gamma^2) \\ -(\beta-\frac{1}{4}\gamma+\frac{\beta^2}{\gamma})(2\beta+\gamma)^2 \end{array}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l}\frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +2\beta^3-\beta\gamma^2+2\beta^2\gamma-\gamma^3 \\ -(\beta-\frac{1}{4}\gamma+\frac{\beta^2}{\gamma})(4\beta^2+4\beta\gamma+\gamma^2)\end{array}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\begin{array}{l}\frac{6\beta^3(\beta+\gamma)}{\gamma} \\ +\frac{5\gamma^3}{8} \\ +2\beta^3-\beta\gamma^2+2\beta^2\gamma-\gamma^3 \\ -4\beta^3-4\beta^2\gamma-\beta\gamma^2 \\ +\beta^2\gamma+\beta\gamma^2+\frac{1}{4}\gamma^3 \\ -\frac{4\beta^4}{\gamma}-4\beta^3-\beta^2\gamma\end{array}\right]$$

$$= \frac{\alpha^2\gamma(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[\frac{2\beta^4}{\gamma}-\frac{1}{8}\gamma^3-2\beta^2\gamma-\beta\gamma^2\right]$$

$$= \frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[2\beta^4-2\beta^2\gamma^2-\beta\gamma^3-\frac{1}{8}\gamma^4\right]$$

$$= \frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(4\beta^2-\gamma^2)^2}\left[2(\beta+\frac{1}{2}\gamma)^2(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)\right]$$

$$= \frac{\alpha^2(\beta-\gamma)}{\beta(\beta+\gamma)(2\beta+\gamma)^2(2\beta-\gamma)^2}\left[(2\beta+\gamma)(\beta+\frac{1}{2}\gamma)(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)\right]$$

$$= \frac{\alpha^2(\beta-\gamma)}{2\beta(\beta+\gamma)(2\beta-\gamma)^2}\left[\underbrace{(\beta+\frac{\sqrt{2}-1}{2}\gamma)}_{>0}(\beta-\frac{\sqrt{2}+1}{2}\gamma)\right]$$

(48) Equation

$$\frac{\partial(TS_{mixed}-TS_{normal})}{\partial\gamma}=\frac{\alpha^2}{2\beta}\partial\frac{(\beta-\gamma)*(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)}{(\beta+\gamma)(2\beta-\gamma)^2}/\partial\gamma$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)+(\beta-\gamma)\left[\frac{\sqrt{2}-1}{2}(\beta-\frac{\sqrt{2}+1}{2}\gamma)-\frac{\sqrt{2}+1}{2}(\beta+\frac{\sqrt{2}-1}{2}\gamma)\right]) \\ *(\beta+\gamma)(2\beta-\gamma)^2 \\ -(\beta-\gamma)(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)\left[(2\beta-\gamma)^2-2(2\beta-\gamma)(\beta+\gamma)\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)+(\beta-\gamma)\left[(\frac{\sqrt{2}-1}{2}\beta-\frac{2-1}{4}\gamma)-(\frac{\sqrt{2}+1}{2}\beta+\frac{2-1}{4}\gamma)\right]) \\ *(\beta+\gamma)(2\beta-\gamma)^2 \\ -(\beta-\gamma)(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)\left[4\beta^2-4\beta\gamma+\gamma^2-(4\beta-2\gamma)(\beta+\gamma)\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)-(\beta-\gamma)(\beta+\frac{1}{2}\gamma))*(\beta+\gamma)(2\beta-\gamma)^2 \\ -(\beta-\gamma)(\beta+\frac{\sqrt{2}-1}{2}\gamma)(\beta-\frac{\sqrt{2}+1}{2}\gamma)\left[4\beta^2-4\beta\gamma+\gamma^2-(4\beta-2\gamma)(\beta+\gamma)\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-(\beta^2-\frac{\sqrt{2}+1}{2}\beta\gamma+\frac{\sqrt{2}-1}{2}\beta\gamma-\frac{\sqrt{2}-1}{2}\frac{\sqrt{2}+1}{2}\gamma^2)-(\beta-\gamma)(\beta+\frac{1}{2}\gamma))(\beta+\gamma)(2\beta-\gamma)^2 \\ -(\beta-\gamma)(\beta^2-\frac{\sqrt{2}+1}{2}\beta\gamma+\frac{\sqrt{2}-1}{2}\beta\gamma-\frac{\sqrt{2}-1}{2}\frac{\sqrt{2}+1}{2}\gamma^2)\left[4\beta^2-4\beta\gamma+\gamma^2-(4\beta-2\gamma)(\beta+\gamma)\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-(\beta^2-\beta\gamma-\frac{1}{4}\gamma^2)-(\beta-\gamma)(\beta+\frac{1}{2}\gamma))(\beta+\gamma)(2\beta-\gamma)^2\\ -(\beta-\gamma)(\beta^2-\beta\gamma-\frac{1}{4}\gamma^2)\left[4\beta^2-4\beta\gamma+\gamma^2-4\beta^2-4\beta\gamma+2\beta\gamma+2\gamma^2\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-\beta^2+\beta\gamma+\frac{1}{4}\gamma^2-\beta^2-\frac{1}{2}\gamma\beta+\beta\gamma+\frac{1}{2}\gamma^2)(\beta+\gamma)(2\beta-\gamma)^2\\ -(\beta-\gamma)(\beta^2-\beta\gamma-\frac{1}{4}\gamma^2)\left[4\beta^2-4\beta\gamma+\gamma^2-4\beta^2-4\beta\gamma+2\beta\gamma+2\gamma^2\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-\beta^2+\beta\gamma+\frac{1}{4}\gamma^2-\beta^2-\frac{1}{2}\gamma\beta+\beta\gamma+\frac{1}{2}\gamma^2)(\beta+\gamma)(2\beta-\gamma)^2\\ -\frac{3}{4}(\beta-\gamma)(4\beta^2-4\beta\gamma-\gamma^2)\left[-2\beta\gamma+\gamma^2\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-2\beta^2+\frac{3}{2}\beta\gamma+\frac{3}{4}\gamma^2)(\beta+\gamma)(2\beta-\gamma)^2\\ -\frac{3}{4}(\beta-\gamma)(4\beta^2-4\beta\gamma-\gamma^2)\left[-2\beta\gamma+\gamma^2\right]\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{2\beta}\frac{\begin{bmatrix}(-2\beta^2+\frac{3}{2}\beta\gamma+\frac{3}{4}\gamma^2)(\beta+\gamma)(2\beta-\gamma)^2\\ +\frac{3}{4}\gamma(\beta-\gamma)(4\beta^2-4\beta\gamma-\gamma^2)(2\beta-\gamma)\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^4}$$

$$=\frac{\alpha^2}{8\beta}\frac{\begin{bmatrix}(-8\beta^2+6\beta\gamma+3\gamma^2)(\beta+\gamma)(2\beta-\gamma)\\ +3\gamma(\beta-\gamma)(4\beta^2-4\beta\gamma-\gamma^2)\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$=\frac{\alpha^2}{8\beta}\frac{\begin{bmatrix}-16\beta^4-8\beta^3\gamma+8\beta^2\gamma^2\\ +12\beta^3\gamma+6\beta^2\gamma^2-6\beta\gamma^3\\ +6\beta^2\gamma^2+3\beta\gamma^3-3\gamma^4\\ +12\beta^3\gamma-12\beta^2\gamma^2-3\beta\gamma^3\\ -12\beta^2\gamma^2+12\beta\gamma^3+3\gamma^4\end{bmatrix}}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$=\frac{\alpha^2}{8\beta}\frac{\left[-16\beta^4+16\beta^3\gamma-4\beta^2\gamma^2+6\beta\gamma^3\right]}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$=\frac{\alpha^2}{4}\frac{\left[-8\beta^3+8\beta^2\gamma-2\beta\gamma^2+3\gamma^3\right]}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= -2\alpha^2 \frac{\left[\beta^3 - \beta^2\gamma + \frac{1}{4}\beta\gamma^2 - \frac{3}{8}\gamma^3\right]}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= -2\alpha^2 \frac{(\beta - \frac{2\sqrt{z}+1}{2}\gamma)(\beta^2 + \frac{2\sqrt{z}-1}{2}\beta\gamma + z\gamma^2) // z \approx 0{,}345}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

$$= -2\alpha^2 \frac{(\beta - 1{,}86\gamma)(\beta^2 + 0{,}86\beta\gamma + {,}345\gamma^2)}{(\beta+\gamma)^2(2\beta-\gamma)^3}$$

(49) Equation

$$\max_{p_y} \pi(p_y, y) = yp_y - c_y y //$$

$$= y(p_y - c_y) // y = \frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x$$

$$= (\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\beta_1\beta_2 - \gamma^2} - \frac{\beta_1}{\beta_1\beta_2 - \gamma^2} p_y + \frac{\gamma}{\beta_1\beta_2 - \gamma^2} p_x)(p_y - c_y) // \beta_1\beta_2 - \gamma^2 = \delta > 0$$

$$= (\frac{\beta_1\alpha_2 - \alpha_1\gamma}{\delta} - \frac{\beta_1}{\delta} p_y + \frac{\gamma}{\delta} p_x)(p_y - c_y) // \alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, c_x = c_y = 0, \delta = \beta^2 - \gamma^2$$

$$= (\frac{\beta\alpha - \alpha\gamma}{\delta} - \frac{\beta}{\delta} p_y + \frac{\gamma}{\delta} p_x)(p_y)$$

$$= \frac{\beta\alpha - \alpha\gamma}{\delta} p_y - \frac{\beta}{\delta} p_y^2 + \frac{\gamma}{\delta} p_x p_y$$

(50) Equation n/a

(51) Equation

$$\max_{p_x} \alpha_1 x + \alpha_2 y - (\beta_1 x^2 + 2\gamma yx + \beta_2 y^2)/2 - p_y y - c_x x // \alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta, c_x = c_y = 0$$

$$= \alpha x + \alpha y - (\beta x^2 + 2\gamma yx + \beta y^2)/2 - p_y y // p_y(p_x) = \frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x$$

$$= \alpha x + \alpha y - (\beta x^2 + 2\gamma yx + \beta y^2)/2 - y(\frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x)$$

$$= \alpha x + \alpha y - (\beta x^2 + 2\gamma yx + \beta y^2)/2 - y(\frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x) // \begin{bmatrix} y = \frac{\beta\alpha - \alpha\gamma}{\delta} - \frac{\beta}{\delta}(\frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x) + \frac{\gamma}{\delta} p_x \\ = \frac{\beta\alpha - \alpha\gamma}{\delta} - \frac{\beta\alpha - \alpha\gamma}{2\delta} + (\frac{\gamma}{\delta} - \frac{\gamma}{2\delta}) p_x \\ = \frac{\beta\alpha - \alpha\gamma}{2\delta} + \frac{\gamma}{2\delta} p_x \\ \frac{\partial y}{\partial p_x} = \frac{\gamma}{2\delta} \\ x = \frac{\beta\alpha - \alpha\gamma}{\delta} - \frac{\beta}{\delta} p_x + \frac{\gamma}{\delta}(\frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x) \\ = \frac{\beta\alpha - \alpha\gamma}{\delta} + \frac{\gamma(\beta\alpha - \alpha\gamma)}{2\beta\delta} + p_x(\frac{\gamma^2}{2\beta\delta} - \frac{\beta}{\delta}) \\ \frac{\partial x}{\partial p_x} = \frac{\gamma^2}{2\beta\delta} - \frac{\beta}{\delta} \\ \delta = \beta^2 - \gamma^2 \end{bmatrix}$$

$F.O.C$

$$\frac{\partial}{\partial p_x} = \alpha(\frac{\partial x}{\partial p_x} + \frac{\partial y}{\partial p_x}) - (2\beta x \frac{\partial x}{\partial p_x} + 2\gamma(\frac{\partial y}{\partial p_x} x + \frac{\partial x}{\partial p_x} y) + 2\beta y \frac{\partial y}{\partial p_x})/2 - \frac{\partial y}{\partial p_x}(\frac{\beta\alpha - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x) - y(\frac{\gamma}{2\beta}) = 0$$

$$\Leftrightarrow \alpha \frac{\partial x}{\partial p_x} + \alpha \frac{\partial y}{\partial p_x} - \beta x \frac{\partial x}{\partial p_x}$$

$$- \gamma \frac{\partial y}{\partial p_x} x - \gamma \frac{\partial x}{\partial p_x} y$$

$$- \beta y \frac{\partial y}{\partial p_x} - \frac{\partial y}{\partial p_x} \frac{\beta\alpha - \alpha\gamma}{2\beta} - \frac{\partial y}{\partial p_x} \frac{\gamma}{2\beta} p_x - y \frac{\gamma}{2\beta} = 0$$

$$\Leftrightarrow$$

$$\alpha(\frac{\gamma^2}{2\beta\delta}-\frac{\beta}{\delta})+\alpha\frac{\gamma}{2\delta}-\beta(\frac{\beta\alpha-\alpha\gamma}{\delta}+\frac{\gamma(\beta\alpha-\alpha\gamma)}{2\beta\delta}+p_x(\frac{\gamma^2}{2\beta\delta}-\frac{\beta}{\delta}))(\frac{\gamma^2}{2\beta\delta}-\frac{\beta}{\delta})$$
$$-\gamma\frac{\gamma}{2\delta}(\frac{\beta\alpha-\alpha\gamma}{\delta}+\frac{\gamma(\beta\alpha-\alpha\gamma)}{2\beta\delta}+p_x(\frac{\gamma^2}{2\beta\delta}-\frac{\beta}{\delta}))-\gamma(\frac{\gamma^2}{2\beta\delta}-\frac{\beta}{\delta})(\frac{\beta\alpha-\alpha\gamma}{2\delta}+\frac{\gamma}{2\delta}p_x)$$
$$-\beta(\frac{\beta\alpha-\alpha\gamma}{2\delta}+\frac{\gamma}{2\delta}p_x)\frac{\gamma}{2\delta}-\frac{\gamma}{2\delta}\frac{\beta\alpha-\alpha\gamma}{2\beta}-\frac{\gamma}{2\delta}\frac{\gamma}{2\beta}p_x-(\frac{\beta\alpha-\alpha\gamma}{2\delta}+\frac{\gamma}{2\delta}p_x)\frac{\gamma}{2\beta}=0$$

$$\Leftrightarrow\frac{\alpha\gamma^2}{2\beta\delta}-\frac{\alpha\beta}{\delta}+\frac{\alpha\gamma}{2\delta}-\frac{(\beta^2\alpha-\beta\alpha\gamma)(\gamma^2-2\beta^2)}{2\beta\delta^2}-\frac{\gamma(\beta\alpha-\alpha\gamma)(\gamma^2-2\beta^2)}{4\beta\delta^2}-p_x(\frac{(\gamma^2-2\beta^2)^2}{4\beta\delta^2})$$
$$-\frac{\gamma^2(\beta\alpha-\alpha\gamma)}{2\delta^2}-\frac{\gamma^3(\beta\alpha-\alpha\gamma)}{4\beta\delta^2}-p_x(\frac{\gamma^4}{4\beta\delta^2}-\frac{\gamma^2\beta}{2\delta^2})-\frac{(\beta\alpha-\alpha\gamma)(\gamma^3-2\gamma\beta^2)}{4\beta\delta^2}-\frac{\gamma(\gamma^3-2\gamma\beta^2)}{4\beta\delta^2}p_x$$
$$-\frac{\beta^2\alpha\gamma-\alpha\beta\gamma^2}{4\delta^2}-\frac{\beta\gamma^2}{4\delta^2}p_x-\frac{\beta\alpha\gamma-\alpha\gamma^2}{4\beta\delta}-\frac{\gamma^2}{4\beta\delta}p_x-\frac{\beta\gamma\alpha-\alpha\gamma^2}{4\beta\delta}-\frac{\gamma^2}{4\beta\delta}p_x=0 // ok$$

$$\Leftrightarrow\frac{\alpha\gamma^2}{2\beta\delta}-\frac{\alpha\beta}{\delta}+\frac{\alpha\gamma}{2\delta}-\frac{(\beta^2\alpha-\beta\alpha\gamma)(\gamma^2-2\beta^2)}{2\beta\delta^2}-\frac{\gamma(\beta\alpha-\alpha\gamma)(\gamma^2-2\beta^2)}{4\beta\delta^2}$$
$$-\frac{\gamma^2(\beta\alpha-\alpha\gamma)}{2\delta^2}-\frac{\gamma^3(\beta\alpha-\alpha\gamma)}{4\beta\delta^2}-\frac{(\beta\alpha-\alpha\gamma)(\gamma^3-2\gamma\beta^2)}{4\beta\delta^2}$$
$$-\frac{\beta^2\alpha\gamma-\alpha\beta\gamma^2}{4\delta^2}-\frac{\beta\alpha\gamma-\alpha\gamma^2}{4\beta\delta}-\frac{\beta\gamma\alpha-\alpha\gamma^2}{4\beta\delta}$$
$$=\frac{\gamma^2}{4\beta\delta}p_x+p_x\frac{(\gamma^2-2\beta^2)^2}{4\beta\delta^2}+\frac{\gamma(\gamma^3-2\gamma\beta^2)}{4\beta\delta^2}p_x+p_x\frac{\gamma^4-2\beta^2\gamma^2}{4\beta\delta^2}+\frac{\beta^2\gamma^2}{4\beta\delta^2}p_x+\frac{\gamma^2}{4\beta\delta}p_x$$

$$\Leftrightarrow \frac{\alpha\gamma^2}{2\beta\delta} - \frac{\alpha\beta}{\delta} + \frac{\alpha\gamma}{2\delta} - \frac{(\beta^2\alpha - \beta\alpha\gamma)(\gamma^2 - 2\beta^2)}{2\beta\delta^2} - \frac{2\gamma(\beta\alpha - \alpha\gamma)(\gamma^2 - 2\beta^2)}{4\beta\delta^2}$$

$$- \frac{\gamma^2(\beta\alpha - \alpha\gamma)}{2\delta^2} - \frac{\gamma^3(\beta\alpha - \alpha\gamma)}{4\beta\delta^2} - \frac{\beta\gamma(\beta\alpha - \alpha\gamma)}{4\delta^2} - \frac{2\gamma(\beta\alpha - \alpha\gamma)}{4\beta\delta}$$

$$= p_x \frac{(\gamma^2 - 2\beta^2)^2}{4\beta\delta^2} + p_x \frac{2\gamma^2(\gamma^2 - 2\beta^2)}{4\beta\delta^2} + \frac{\beta^2\gamma^2}{4\beta\delta^2} p_x + \frac{2\gamma^2\delta}{4\beta\delta^2} p_x$$

$$\Leftrightarrow \frac{\alpha\gamma^2}{2\beta\delta} - \frac{\alpha\beta}{\delta} + \frac{\alpha\gamma}{2\delta}$$

$$- \frac{(\beta^2\alpha - \beta\alpha\gamma)(\gamma^2 - 2\beta^2)}{2\beta\delta^2} - \frac{2\gamma(\beta\alpha - \alpha\gamma)(\gamma^2 - 2\beta^2)}{4\beta\delta^2}$$

$$- \frac{\gamma^2(\beta\alpha - \alpha\gamma)}{2\delta^2} - \frac{\gamma^3(\beta\alpha - \alpha\gamma)}{4\beta\delta^2} - \frac{\beta\gamma(\beta\alpha - \alpha\gamma)}{4\delta^2} - \frac{2\gamma\delta(\beta\alpha - \alpha\gamma)}{4\beta\delta^2}$$

$$= p_x \frac{\left((\gamma^2 - 2\beta^2)^2 + 2\gamma^2(\gamma^2 - 2\beta^2) + \beta^2\gamma^2 + 2\gamma^2\delta\right)}{4\beta\delta^2}$$

$$\Leftrightarrow p_x\left(3\gamma^4 - 7\beta^2\gamma^2 + 4\beta^4 + 2\gamma^2\delta\right) = \alpha \begin{bmatrix} 2\delta\gamma^2 - 4\beta^2\delta + 2\beta\delta\gamma \\ -2\beta(\beta-\gamma)(\gamma^2 - 2\beta^2) - 2\gamma(\beta-\gamma)(\gamma^2 - 2\beta^2) \\ -2\beta\gamma^2(\beta-\gamma) - \gamma^3(\beta-\gamma) - \beta^2\gamma(\beta-\gamma) - 2\gamma\delta(\beta-\gamma) \end{bmatrix}$$

$\Leftrightarrow$

$$p_x = \frac{\alpha}{3\gamma^4 - 7\beta^2\gamma^2 + 4\beta^4 + 2\gamma^2\delta} \begin{bmatrix} 2\delta\gamma^2 - 4\beta^2\delta + 2\beta\delta\gamma \\ -2(\beta+\gamma)(\beta-\gamma)(\gamma^2 - 2\beta^2) \\ -2\beta\gamma^2(\beta-\gamma) - \gamma^3(\beta-\gamma) - \beta^2\gamma(\beta-\gamma) - 2\gamma\delta(\beta-\gamma) \end{bmatrix} // \delta = (\beta-\gamma)(\beta+\gamma)$$

$$p_x = \frac{\alpha(\beta-\gamma)}{3\gamma^4 - 7\beta^2\gamma^2 + 4\beta^4 + 2\gamma^2(\beta^2-\gamma^2)} \begin{bmatrix} 2(\beta+\gamma)\gamma^2 - 4\beta^2(\beta+\gamma) + 2\beta(\beta+\gamma)\gamma \\ -2(\beta+\gamma)(\gamma^2 - 2\beta^2) \\ -2\beta\gamma^2 - \gamma^3 - \beta^2\gamma - 2\gamma(\beta+\gamma)(\beta-\gamma) \end{bmatrix}$$

$$\Leftrightarrow p_x = \frac{\alpha(\beta-\gamma)}{3\gamma^4 - 7\beta^2\gamma^2 + 4\beta^4 + 2\gamma^2\beta^2 - 2\gamma^4} \begin{bmatrix} (\beta+\gamma)(2\gamma^2 - 4\beta^2 + 2\beta\gamma - 2\gamma^2 + 4\beta^2 - 2\beta\gamma + 2\gamma^2) \\ -2\beta\gamma^2 - \gamma^3 - \beta^2\gamma \end{bmatrix}$$

$$\Leftrightarrow p_x = \frac{\alpha(\beta-\gamma)}{\gamma^4 - 5\beta^2\gamma^2 + 4\beta^4} \begin{bmatrix} (\beta+\gamma)2\gamma^2 \\ -2\beta\gamma^2 - \gamma^3 - \beta^2\gamma \end{bmatrix}$$

$$\Leftrightarrow p_x = \frac{\alpha(\beta-\gamma)}{\gamma^4 - 5\beta^2\gamma^2 + 4\beta^4} \left[2\beta\gamma^2 + 2\gamma^3 - 2\beta\gamma^2 - \gamma^3 - \beta^2\gamma\right]$$

$$\Leftrightarrow p_x = \frac{\alpha(\beta-\gamma)}{\gamma^4 - 5\beta^2\gamma^2 + 4\beta^4} \left[\gamma^3 - \beta^2\gamma\right]$$

$$\Leftrightarrow p_x = \frac{\alpha\gamma(\beta-\gamma)(\gamma+\beta)}{\gamma^4 - 5\beta^2\gamma^2 + 4\beta^4} \left[(\gamma-\beta)\right]$$

$$\Leftrightarrow p_x = \frac{\alpha\gamma(\beta-\gamma)(\gamma+\beta)}{(\gamma-\beta)(\gamma-2\beta)(\gamma+\beta)(\gamma+2\beta)} \left[(\gamma-\beta)\right]$$

$$\Leftrightarrow p_x = \frac{\alpha\gamma(\beta-\gamma)}{(\gamma+2\beta)\underbrace{(\gamma-2\beta)}_{\beta>\gamma\to<0}} < 0$$

(52) Equation

$$p_x = \frac{\alpha\gamma(\beta-\gamma)}{(\gamma+2\beta)\underbrace{(\gamma-2\beta)}_{\beta>\gamma\rightarrow<0}} < 0$$

$$RC_y = p_y(p_x) = \frac{\beta\alpha-\alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_x \,//\, p_x = \frac{\alpha\gamma(\beta-\gamma)}{(\gamma+2\beta)(\gamma-2\beta)}$$

$$\Leftrightarrow p_y(p_x) = \frac{\alpha}{2\beta}(\beta-\gamma) + \frac{\gamma}{2\beta}\frac{\alpha\gamma(\beta-\gamma)}{(\gamma+2\beta)(\gamma-2\beta)}$$

$$= \frac{\alpha}{2\beta}(\beta-\gamma) + \frac{\alpha}{2\beta}\frac{\gamma^2(\beta-\gamma)}{(\gamma+2\beta)(\gamma-2\beta)}$$

$$= \frac{\alpha(\beta-\gamma)}{2\beta}\left[1 + \frac{\gamma^2}{(\gamma+2\beta)(\gamma-2\beta)}\right]$$

$$= \frac{\alpha(\beta-\gamma)}{2\beta}\left[\frac{(\gamma+2\beta)(\gamma-2\beta)}{(\gamma+2\beta)(\gamma-2\beta)} + \frac{\gamma^2}{(\gamma+2\beta)(\gamma-2\beta)}\right]$$

$$= \frac{\alpha(\beta-\gamma)}{2\beta}\left[\frac{\gamma^2-4\beta^2}{(\gamma+2\beta)(\gamma-2\beta)} + \frac{\gamma^2}{(\gamma+2\beta)(\gamma-2\beta)}\right]$$

$$= \frac{\alpha(\beta-\gamma)}{2\beta}\left[\frac{2\gamma^2-4\beta^2}{(\gamma+2\beta)(\gamma-2\beta)}\right]$$

$$= \frac{\alpha(\beta-\gamma)}{\beta}\left[\frac{\gamma^2-2\beta^2}{(\gamma+2\beta)(\gamma-2\beta)}\right]$$

$$= \frac{\alpha(\beta-\gamma)}{\beta}\left[\frac{2\beta^2-\gamma^2}{(\gamma+2\beta)(2\beta-\gamma)}\right] > 0$$

(53) Equation

$$\overbrace{\frac{\alpha(\beta-\gamma)}{\beta}\left[\frac{2\beta^2-\gamma^2}{(\gamma+2\beta)(2\beta-\gamma)}\right]}^{Stackelberg} < \overbrace{\frac{\alpha(\beta-\gamma)}{2\beta}}^{Bertrand}$$

$$\Leftrightarrow \frac{2\beta^2-\gamma^2}{(\gamma+2\beta)(2\beta-\gamma)} < \frac{1}{2}$$

$$\Leftrightarrow \frac{2\beta^2-\gamma^2}{4\beta^2-\gamma^2} < \frac{1}{2}$$

$$\Leftrightarrow 2\beta^2-\gamma^2 < 2\beta^2-\frac{1}{2}\gamma^2$$

$$\Leftrightarrow 0 < \frac{1}{2}\gamma^2$$

q.e.d

(54) Equation n/a

(55) Equation n/a

(56) Equation

$$F.O.C \quad (members)$$

$$\frac{\partial U_m}{\partial x_m} = \alpha - \beta x_m - \gamma y_m = 0 \Leftrightarrow x_m = \frac{\alpha - \gamma y_m}{\beta}$$

$$\frac{\partial U_m}{\partial y_m} = \alpha - \beta y_m - \gamma x_m - p_y = 0 \Leftrightarrow p_y = \alpha - \beta y_m - \gamma x_m$$

$$p_y = \alpha - \beta y_m - \gamma x_m \; // \; x_m = \frac{\alpha - \gamma y_m}{\beta}$$

$$\Leftrightarrow p_y = \alpha - \beta y_m - \frac{\gamma\alpha}{\beta} + \frac{\gamma^2 y_m}{\beta} \; //* \beta$$

$$\Leftrightarrow \beta p_y = \alpha\beta - \beta^2 y_m - \alpha\gamma + \gamma^2 y_m$$

$$\Leftrightarrow \beta^2 y_m - \gamma^2 y_m = \alpha\beta - \alpha\gamma - \beta p_y$$

$$\Leftrightarrow y_m = \frac{\alpha\beta - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y$$

$$x_m = \frac{\alpha}{\beta} - \frac{\gamma}{\beta} y_m \; // \; y_m = \frac{\alpha\beta - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y$$

$$\Leftrightarrow x_m = \frac{\alpha}{\beta} - \frac{\gamma(\alpha\beta - \alpha\gamma)}{\beta(\beta^2 - \gamma^2)} + \frac{\gamma}{(\beta^2 - \gamma^2)} p_y$$

$$\Leftrightarrow x_m = \frac{\alpha\beta^2 - \alpha\gamma^2 - \alpha\gamma\beta + \alpha\gamma^2}{\beta(\beta^2 - \gamma^2)} + \frac{\gamma}{(\beta^2 - \gamma^2)} p_y$$

$$\Leftrightarrow x_m = \frac{\alpha\beta - \alpha\gamma}{\beta^2 - \gamma^2} + \frac{\gamma}{\beta^2 - \gamma^2} p_y$$

(57) Equation

$$F.O.C \quad (non-members)$$

$$y_r = \frac{\beta\alpha - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y + \frac{\gamma}{\beta^2 - \gamma^2} p_x$$

$$x_r = \frac{\beta\alpha - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_x + \frac{\gamma}{\beta^2 - \gamma^2} p_y$$

(58) Equation

$$\max_{p_y} \pi = p_y y_m + p_y y_r$$

$$= p_y(y_m + y_r) // \left| \begin{array}{l} y_r = \frac{\beta\alpha - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y + \frac{\gamma}{\beta^2 - \gamma^2} p_x \\ y_m = \frac{\alpha\beta - \alpha\gamma}{\beta^2 - \gamma^2} - \frac{\beta}{\beta^2 - \gamma^2} p_y \end{array} \right|$$

$$= p_y \left(\frac{2(\alpha\beta - \alpha\gamma)}{\beta^2 - \gamma^2} - \frac{2\beta}{\beta^2 - \gamma^2} p_y + \frac{\gamma}{\beta^2 - \gamma^2} p_x \right)$$

$F.O.C$

$$\frac{\partial \pi}{\partial p_y} = \frac{2(\alpha\beta - \alpha\gamma)}{\beta^2 - \gamma^2} - \frac{2\beta}{\beta^2 - \gamma^2} p_y + \frac{\gamma}{\beta^2 - \gamma^2} p_x - \frac{2\beta}{\beta^2 - \gamma^2} p_y = 0$$

$$\Leftrightarrow \frac{4\beta}{\beta^2 - \gamma^2} p_y = \frac{2(\alpha\beta - \alpha\gamma)}{\beta^2 - \gamma^2} + \frac{\gamma}{\beta^2 - \gamma^2} p_x$$

$$\Leftrightarrow 4\beta p_y = 2(\alpha\beta - \alpha\gamma) + \gamma p_x$$

$$\Leftrightarrow p_y(p_x) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{4\beta} p_x$$

(59) Equation

$$\max_{p_x} U_m = \alpha x_m + \alpha y_m - \tfrac{1}{2}(\beta x_m^2 + 2\gamma x_m y_m + \beta y_m^2) - p_y y_m + \frac{(1-a)}{a} x_r p_x \; // \left| \begin{array}{l} x_m = \dfrac{\alpha\beta - \alpha\gamma}{\delta} + \dfrac{\gamma}{\delta} p_y \\ \dfrac{\partial x_m}{\partial p_x} = 0 \\ x_r = \dfrac{\beta\alpha - \alpha\gamma}{\delta} - \dfrac{\beta}{\delta} p_x + \dfrac{\gamma}{\delta} p_y \\ \dfrac{\partial x_r}{\partial p_x} = -\dfrac{\beta}{\delta} \\ y_m = \dfrac{\alpha\beta - \alpha\gamma}{\delta} - \dfrac{\beta}{\delta} p_y \\ \dfrac{\partial y_m}{\partial p_x} = 0 \\ \delta = \beta^2 - \gamma^2 \end{array} \right]$$

$F.O.C$

$$\frac{\partial U_m}{\partial p_x} = \alpha \frac{\partial x_m}{\partial p_x} + \alpha \frac{\partial y_m}{\partial p_x} - \tfrac{1}{2}(2\beta x_m \frac{\partial x_m}{\partial p_x} + 2\gamma(x_m \frac{\partial y_m}{\partial p_x} + y_m \frac{\partial x_m}{\partial p_x}) + 2\beta y_m \frac{\partial y_m}{\partial p_x})$$

$$- p_y \frac{\partial y_m}{\partial p_x} + \frac{(1-a)}{a}\frac{\partial x_r}{\partial p_x} p_x + \frac{(1-a)}{a} x_r = 0$$

$$\Leftrightarrow \frac{(1-a)}{a} x_r = \frac{(1-a)}{a}\frac{\beta}{\beta^2 - \gamma^2} p_x$$

$$\Leftrightarrow \frac{\beta\alpha - \alpha\gamma}{\delta} - \frac{\beta}{\delta} p_x + \frac{\gamma}{\delta} p_y = \frac{\beta}{\beta^2 - \gamma^2} p_x$$

$$\Leftrightarrow \beta\alpha - \alpha\gamma + \gamma p_y = 2\beta p_x$$

$$\Leftrightarrow p_x(p_y) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_y$$

(60) Equation

$$p_x(p_y) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta} p_y \,//\, p_y(p_x) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{4\beta} p_x$$

$$= \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{2\beta}\left(\frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{4\beta} p_x\right)$$

$$= \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\alpha\beta\gamma - \alpha\gamma^2}{4\beta^2} + \frac{\gamma^2}{8\beta^2} p_x$$

$$\Leftrightarrow p_x = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\alpha\beta\gamma - \alpha\gamma^2}{4\beta^2} + \frac{\gamma^2}{8\beta^2} p_x$$

$$\Leftrightarrow p_x\left(\frac{8\beta^2 - \gamma^2}{8\beta^2}\right) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\alpha\beta\gamma - \alpha\gamma^2}{4\beta^2}$$

$$\Leftrightarrow p_x = \frac{4\beta(\alpha\beta - \alpha\gamma) + 2\gamma(\alpha\beta - \alpha\gamma)}{8\beta^2 - \gamma^2}$$

$$\Leftrightarrow p_x^N = \frac{2\alpha(\beta - \gamma)(2\beta + \gamma)}{8\beta^2 - \gamma^2}$$

$$p_y(p_x) = \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\gamma}{4\beta} p_x \,//\, p_x = \frac{2\alpha(\beta - \gamma)(2\beta + \gamma)}{8\beta^2 - \gamma^2}$$

$$= \frac{\alpha\beta - \alpha\gamma}{2\beta} + \frac{\alpha\gamma(\beta - \gamma)(2\beta + \gamma)}{2\beta(8\beta^2 - \gamma^2)}$$

$$= \frac{\alpha(\beta - \gamma)(8\beta^2 - \gamma^2)}{2\beta(8\beta^2 - \gamma^2)} + \frac{\alpha\gamma(\beta - \gamma)(2\beta + \gamma)}{2\beta(8\beta^2 - \gamma^2)}$$

$$= \frac{\alpha(\beta - \gamma)(8\beta^2 + 2\beta\gamma + \gamma^2 - \gamma^2)}{2\beta(8\beta^2 - \gamma^2)}$$

$$\Leftrightarrow p_y^N = \frac{\alpha(\beta - \gamma)(4\beta + \gamma)}{(8\beta^2 - \gamma^2)}$$

(61) Equation

$$\overbrace{\frac{2\alpha(\beta-\gamma)(2\beta+\frac{1}{2}\gamma)}{8\beta^2-\gamma^2}}^{mixed\ duopoly:\ cooperative\ sales\ also\ to\ non-members} > \overbrace{\frac{\beta\alpha-\alpha\gamma}{2\beta}}^{mixed\ duopoly:\ cooperative\ sales\ only\ to\ members}$$

$$\Leftrightarrow \frac{4\beta(2\beta+\frac{1}{2}\gamma)}{8\beta^2-\gamma^2} > 1$$

$$\Leftrightarrow 8\beta^2+2\beta\gamma > 8\beta^2-\gamma^2$$

$$\Leftrightarrow 2\beta\gamma > -\gamma^2$$

$$\Leftrightarrow 2\beta\gamma+\gamma^2 > 0$$

q.e.d.

(62) Equation

$$\overbrace{\frac{2\alpha(\beta-\gamma)(\beta+\frac{1}{2}\gamma)}{4\beta^2-\gamma^2}}^{normal\ duopoly} > \overbrace{\frac{2\alpha(\beta-\gamma)(2\beta+\frac{1}{2}\gamma)}{8\beta^2-\gamma^2}}^{mixed\ duopoly:\ cooperative\ sales\ also\ to\ non-members}$$

$$\Leftrightarrow \frac{(\beta+\frac{1}{2}\gamma)}{4\beta^2-\gamma^2} > \frac{(2\beta+\frac{1}{2}\gamma)}{8\beta^2-\gamma^2}$$

$$\Leftrightarrow (\beta+\frac{1}{2}\gamma)(8\beta^2-\gamma^2) > (2\beta+\frac{1}{2}\gamma)(4\beta^2-\gamma^2)$$

$$\Leftrightarrow (\beta+\frac{1}{2}\gamma)(8\beta^2-\gamma^2) > (2\beta+\frac{1}{2}\gamma)(2\beta+\gamma)(2\beta-\gamma)$$

$$\Leftrightarrow \frac{1}{2}(2\beta+\gamma)(8\beta^2-\gamma^2) > (2\beta+\frac{1}{2}\gamma)(2\beta+\gamma)(2\beta-\gamma)$$

$$\Leftrightarrow \frac{1}{2}(8\beta^2-\gamma^2) > (2\beta+\frac{1}{2}\gamma)(2\beta-\gamma)$$

$$\Leftrightarrow 4\beta^2-\frac{1}{2}\gamma^2 > 4\beta^2-2\beta\gamma+\frac{1}{2}\beta\gamma-\frac{1}{2}\gamma^2$$

$$\Leftrightarrow 0 > -\frac{3}{2}\beta\gamma$$

q.e.d

(63) Equation n/a