Aalto University
School of Economics

Macroeconomics in interest rate term structure modelling

Economics

Master's thesis

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TIIVISTELMÄ

Tässä Pro gradu –tutkielmassa käytetty lähestymistapa on kirjallisuuskatsaus, joka kuvaa korkorakennemallien viimeisimpiä uudistuksia. Painopisteenä ovat korkorakennemallit, joissa käytetään makrotaloudellisia malleja tai muuttujia selittävinä tekijöinä. Tämä tapa tulkita korkorakennetta on tullut vallitsevaksi lähestymistavaksi 2000-luvulla. Siitä huolimatta aiheesta ei ole tehty kattavaa kirjallisuuskatsausta, vaikka ko. mallit ovat kehittyneet nopeasti.

Vasta viimeisimmän vuosikymmenen aikana on korkorakennetta pyritty selittämään makromuuttujien avulla. Tätä kuitenkin edeltää pitkä tutkimusperinne, jossa painopisteenä on ollut matematiikka. Eli mitään ”todellista” selittävää tekijää ei ole käytetty. Tämän hetkinen tutkimus on osoittanut, että makrotaloudelliset muuttujat ovat tärkeä osa korkojen muodostumista. Tätä puoltaa niin teoria kuin empiiriset tulokset.

Korkorakenne on ollut hyvin suosittu tutkimusaihe jo useita vuosikymmeniä, siksi lähdemateriaalia valinta ei ole ollut itsestään selvää. Käytyäni läpi suuren määrän mahdollista lähdeaineistoa seuraavat kolme melko uutta tutkimusta nousi ylitse muiden Hördahl ym. (2006), De Graeve ym. (2009) sekä De Grauwe (2008).

Kirjallisuuskatsauksen ydin on jaettu kolmeen eri osaan: VAR-malliin, DSGE malliin ja kolmeen rakenneyhtälömalliin. Lähes kaikki tarkemmin esitellyt ja analysoidut mallit omaavat myös empiirisen osuuden. Näitä eri mallien empiirisiä tuloksia on verrattu keskenään. Tämä keskinäinen vertailu osoitti, että vain muutama malli tuottaa empiirisesti tarkempia tuloksia kuin satunnaiskulku (random walk) hypoteesi. DSGE-malli ei tuottanut yhtä hyviä tuloksia kuin satunnaiskulku, mutta tämä mallinnusmuoto on teoreettisesti muita malleja parempi sekä erittäin uusi. Siltä voidaan odottaa lähitulevaisuudessa vielä paljon.

Tämä kirjallisuuskatsaus toteaa, että useita ongelmia on ratkaistu, niin siitä huolimatta useita haasteita on yhä olemassa, kuten korkopapereiden tuottojen ja tilamuuttujien välinen lineaarinen riippuvuus. Tämän hetkisen tiedon pohjalta voidaan kuitenkin varmasti sanoa, että makromuuttujilla on huomattava vaikutus korkorakenteeseen. Tästä syystä makromuuttujien pitäisi myös jatkossa olla hyvin merkitsevä osa korkorakenteen mallinnusta.

Avainsanat: korkorakenne, lineaarinen tila-avaruus-malli, VAR-malli, DSGE-malli, affiinikuvaus, Taylor malli ja Uus-Keynes malli.

ABSTRACT

This thesis uses a literature review to describe the latest progress in the models used to describe interest rate term structure (TS). The thesis emphasises macroeconomic variables and models, as these elements have become an essential part of TS modelling in the 21^{st} century. Although this type of literature review has not been conducted before, the rapid development of numerous kinds of models to describe the TS has highlighted the need for a comprehensive summary.

The macroeconomic explanation of TS is quite different than what it used to be. In the past, TS had no real explanation for the driving forces behind the term structure of interest rates. Current research has changed this setup significantly for the better, as the empirical results show that these new models outperform the ones used before the use of macro variables as explanatory variables of TS.

There has been a huge amount of literature on TS, both past and present. Accordingly, the selection of source material for this literature review has not been self-evident. Hördahl et al. (2006), De Graeve et al. (2009) and De Grauwe (2008) are the most interesting recent studies on TS.

The core analysis of this literature review has been divided into three subsections: the VAR model, three structural models and one DSGE model. Almost all of these models contain empirical sections and these have been compared with each other. This comparison showed that only few of these models were better than the random walk hypothesis. Although the DSGE model did not beat the random walk hypothesis, it did give a valid theoretical model that will probably become the starting point for TS in the future.

The result of this literature review is that, although many obstacles have been overcome, several still exist, such as linear dependence between the yields and state variables. The undeniable fact, proven by several recent studies (some of which have been presented in this thesis), is that macro variables have a significant effect on TS. Accordingly, these macro variables should be taken as explanatory variables in the determination of TS.

Keywords: term structure of interest rates, linear state space model, VAR model, DSGE model, affinity, Taylor rule and New Keynesian macro model.

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1 INTRODUCTION

1.1 Background and motivation

The term structure (TS) and forecasting of interest rates are among the most followed pieces of economic information in the world. They have meaningful consequences for almost everyone whose decisions have longer-run economic consequences, such as investment decisions and economic policy. As a result, TS has long been one of the most popular fields of academic study. Financial economists, market participants, central banks and macroeconomists have conducted research to find a more accurate TS model. Hence, it is meaningful to have a comprehensive literature review on the modelling aspects of TS, with the main emphasis on the use of macroeconomic models, which has become a widely studied aspect of TS since the end of the 20th century.

The term structure of an interest rate is a relationship between the time to maturity and the yield to maturity (YTM)[1] of a default-free zero-coupon bond at a given point of time. The optimal series of zero-coupon bonds vary only in maturity, not in other aspects such as risk. This relationship is usually characterised by a zero-coupon yield curve but forward rates and discount function have also been used. As liquid zero-coupon bonds rarely exceed one year in maturity, the bootstrapping technique[2] has been used in order to transform the coupon-bearing bonds and/or the coupons to equivalent zero-coupon bonds.

The ultimate goal of the TS is to demonstrate how the riskless interest rate will vary in its maturity spectrum. This information simplifies decisions for actors ranging from households to firms, while central banks also benefit from this information. The information also helps households decide whether to choose a varying or fixed interest rate for their mortgage.[3] Knowledge of TS helps firms choose the most profitable investment decisions, as well as to decide between a variable and a fixed interest rate for their new debt. In other words, even a

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[1] Yield to maturity is simply an investors' expected return assuming that interest rates will not change before maturity; it is also assumed that the bond issuer will not default. This yield is usually given as a percentage rate per year.

[2] Identical zero coupon bonds for every maturity do not exist, especially for longer maturities. Therefore, the yield needs to be interpolated from the existing market data. This means that each coupon and the face value of coupon bearing bond can be seen as a separate zero coupon bond. See Watsham & Parramore (1997) for concrete examples.

[3] It is quite obvious that banks would, on average, have better forecasts of future interest rate development and would therefore make this decision between varying and fixed mortgage interest rate irrelevant for the households, especially in the long run.

good proxy for a long-run would make the decision-making process much easier when the cost of money in the future can be identified fairly accurately beforehand. There are, of course, all kinds of derivative instruments, especially for firms and larger institutions, which may be used in order to limit risk or make contracts with financial institutions for future lending. In both of these cases, however, hedging might be too expensive relative to the gains and, in the contract case, even unsure, as banks and other financial institutions usually include a clause that gives them the right not to loan in case the borrower is in economic distress. In addition, more accurate TS knowledge enables financial institutions to price many of their derivatives more accurately; naturally, the same applies for the other parties involved, the buyers. For central banks, knowledge of the dynamics between the short and long interest rate makes it easier to adjust policy rates in order to stabilise the economy.

There are several difficulties with an empirically precise TS model. Firstly, one must use the bootstrapping method in order to have estimations of the TS that are longer than one year and, because the bootstrapping itself is estimation, the amount of inaccuracy accumulates. Secondly, on the level that is the actual value, slope denotes how steep a given section of the yield curve is, and curvature describes the shape of the yield curve, whether it is descending, flat or ascending. Each of these three factors of TS varies in time.[4] The third obstacle is how well justified is the assumption of the arbitrage-free framework, which is often used as one of the assumptions in the TS modelling that incorporates macro variables. These TS models that incorporate macro variables will henceforth be abbreviated as MTS.

The absence of arbitrage has significant effects on the yield curve. The dynamic process of the yield curve at any point of time after the needed risk adjustments must be consistent through time with the shape of the yield curve (Diebold et al. 2005). This means that, after determining the risk factors, the yield curve should have the same shape if the given risk factors are the same. This restriction, like any other, leads to less accurate empirical performance if the underlying model is mis-specified (Diebold et al. 2005).

The most widely used TS analysis framework is time series, for which continuous time and discrete time have been used. According to Cochrane (2001), both definitions of time are

[4] One of the first studies to use slope, level and curvature as the names of the factors was Litterman & Scheinkman (1991).

justifiable ways to model TS. Most of the MTS models are affine in yields[5] and nonlinear; models that are non-affine in yields are rare.[6] These non-affine models have probably not been used in the TS modelling that incorporates macroeconomic variables. However, because the generalisation about affinity in yields has gained quite a lot of critique, it is possible that non-affine models will emerge in the near future in addition to the MTS models.

In most of the TS that use continuous time, Brownian motion[7] is used to describe the stochastic nature of the modelling. One of the first to use this stochastic nature in an economic context was Merton, in his 1973 paper "Theory of rational option pricing". Merton used the Brownian motion to describe riskless interest rate, which led to growing interest in models that constructed the TS as a stochastic differential equation. Since then, Merton's work has been seen as a significant part of the TS modelling.

1.2 Research problem and objective of the thesis

This thesis consists of a literature review to the intriguing area of TS modelling via macroeconomic models or variables. The TS model construction was out of the economic context between the 1970s and the late 1990s. It is only in the last decade or so that the most interesting developments in the TS modelling have incorporated the macroeconomic models to TS determination. This interest in a macroeconomic explanation of TS has become the most interesting subsection of TS modelling, and this has also been supported by empirical results, such as the model of Hördahl et al. (2006), in which forecast ability outperforms the random walk hypothesis.[8] However interesting this relationship between the macro variables and TS is, there has been no comprehensive literature review of the various ways in which TS may be modelled via macroeconomic models. Thus, the aim of this thesis is to provide an extensive representation of the latest developments in the area of TS modelling, of which the macro variables and models have become an inseparable element. This literature review will

[5] Affine yields are linear with a constant.
[6] A recent study by Lemke entitled "Threshold dynamics of short-term interest rates: Empirical evidence and implications for the term structure" (2007) for the Deutsche Bundesbank, delivered some empirical and theoretical evidence that these affine models might not be a realistic simplification of TS.
[7] This method was discovered by Robert Brown in the early 19th century while observing the motion of pollen grains (Platen & Heath, 2006). A more precise definition is demonstrated later.
[8] Historically, TS models have found it difficult to beat the random walk hypothesis (Duffee, 2002).

help readers gain a better understanding of how TS has been modelled and where it might evolve in the near future.

The thesis begins with a short presentation of TS modelling history in order to provide a clearer view on how the TS models have evolved over time. It becomes clear in the subsequent chapters that the most intriguing question in MTS models currently is how the macroeconomy is modelled. Many economically-oriented newspapers and magazines, such as Financial Times and The Economist, have questioned the theoretical background of New Classical and New Keynesian macro theory, upon which all TS models that are related to macro economics rely, except the newest comer, the DSGE model. The DSGE model is, in a way, a mixture of these two theories. It is quite probable, therefore, that the variety of macroeconomic models used in TS modelling will continue to increase in the near future. This process will probably be enhanced by the economical downturn, which has historically generated new macroeconomic models.

Since Vasicek's seminal paper was published in the *Journal of Financial Economics* in 1977, there has been a constant process in order to develop more reliable TS model, which would be able to more accurately forecast future interest rate movements. The Vasicek model, as well as the other well known model by Cox et al. (1985), assumed that the instantaneous short rate[9] could be modelled by using the past values of interest rates and some statistical properties. Indeed, these early models of TS can be seen as the beginning of TS modelling as it is seen today. Progress towards achieving a more reliable and accurate model for TS has been gradual but persistent.

The data in the current MTS models is somewhat scattered when it comes to macroeconomic modelling. This is to be expected, given that these MTS models are relatively new. However, as will be explained later, there has been some serious criticism of the most commonly used macromodels, notably against the New Classical and New Keynesian macroeconomic theory. Many of the developed MTS models are somehow related to the ECB, the Bank of England or the Fed and one might assume that these instances would have quite similar macro models to rely on. This is not entirely the case, however, because the ECB has been using, among

[9] In this context, instantaneous interest rate refers to very short term; i.e. time to maturity approaches zero (Vasicek, 1977).

others, the so called Smets-Wounters model,[10] especially for long-run forecasting of the whole Euro-area, while it seems that the Fed has been relying on more “traditional” models. It is clear that these two central banks both use several different types of models for different tasks, although the Smets-Wounters model can be seen as one of the significant differences between the two major central banks (as well as their independence and goals of monetary policy). In other words, as long as the opinion of the most accurate macro model varies between scholars and policy makers,[11] there will be room and resources for new perspectives. It is no surprise, therefore, that the main difference between different MTS models is in the way the macroeconomy has been modelled. There are only minor differences in the actual bond pricing procedure and the combining process of the TS with the macroeconomic model is quite similar between various MTS models.

2 BACKGROUND OF THE TERM STRUCTURE MODELS

2.1 Basic term structure theories

Historically speaking, the various models are based on three distinctive leading TS theories: expectations, liquidity preference, and hedging-pressure.[12] All of these theories have a different interpretation of the leading factors that affect TS determination and each theory also has several subgroups. At this point, however, a robust presentation of these theories is adequate as a reminder of possible ways to interpret TS.

The expectation theory deserves to be the first one presented because it is one of the best known and used theories. Like the other TS theories, the expectation theory has several different variations, which smooth the path for the following interpretations of TS. This basically means that many models require some specific assumptions in order for the model structure to be formally valid, which could lead to some rather illogical conclusions.

2.1.1 Expectations hypothesis

As the name implies, this theory is closely related to expectations. The term is rather loose and there are several sub-categories under this theory but, for the sake of brevity, only one general case will be presented here.

[10] This model, as well as a related working paper by Smets and Wounters, can be found from the ECB’s Internet page http://www.ecb.int/home/html/researcher_swm.en.html.

[11] Each national economy has their own specified characteristics, e.g. the structure of different industries and services and dependence on exports. Excluding these significant differences, however, the actual model should be quite similar between so-called Western economies.

[12] This hedging-pressure theory is also known as preferred habitat theory.

Assuming that investors believe that interest rates are currently too high and that they expect the rate to fall in the near future, longer term bond issues would seem to be more profitable than shorter ones, given that the yield difference between the short- and long-term bonds is low enough to permit a profit-making opportunity. Therefore, long-term bond prices would rise if expectations about lower future interest rates came to fruition. In this situation, short-term bond prices would decrease – that is, increase their yield – while long-term bond prices would increase, thereby closing the gap. In other words, this type of situation could not occur in the absence of arbitrage, where the market participants would trade off even the smallest possible imbalance between the short- and long-term yields. Under the pure expectation hypothesis, the expected return on short- and long-term bonds is zero given the same time horizon, while the looser expectation hypothesis allows the expected return of different maturities to differ only by a constant but not in time (Campbell J. Y., 1995).

The foundation for the expectation hypothesis was originally developed by Hicks (1939) & Lutz (1940), which makes it equally as old as the liquidity preference theory, which was also developed by Hicks (1939). More recent studies of the expectation hypothesis include those by Cox et al. (1981).

The perfect-certainty variant of the expectation hypothesis involves several restrictions that are quite often assumed to be the case. These expectations are that all bonds are default-free, that there are no transaction costs, that all (or at least most) investors make identical and accurate forecasts of future interest rates and that investors are only interested in profit maximisation. These restrictions imply that the long rate is an average of the present and expected short rates (Malkiel, 1966).

This composition of TS has attracted a lot of criticism due to its initial assumptions, namely that the excess returns are neither zero nor time-invariant. Most academics currently build their models under other assumptions, such as time-varying risk premia (Campbell J. Y., 1995).

2.1.2 Liquidity-preference theory

One way to look at bond price determination is through supply and demand factors. For the lender, shorter maturity means that there is likely to be less unwanted news during maturity.

The shorter the maturity, the more liquid the bond is likely to be. For the borrower, however, the opposite is desirable, with a longer need for capital usually being preferred over a short one. Accordingly, there is an imbalance between supply and demand that causes the ascending yield curve; in other words, there should be a risk premium in the longer-term bonds that pushes the yields of long-term bonds upward, which might even be valid with falling future yield expectations.

Critics of the excess return hypothesis, such as Campbell (1995), have noted that the ascending yield curve hypothesis (that the longer the bond maturity, the higher the excess return) has not gained empirical support.

2.1.3 Preferred habitat theory

The preferred habitat theory has been developed as an alternative for the expectations theory. The liquidity-preference theory can be seen as extending the original expectations hypothesis by adding the risk and the demand side to the original expectations theory. The main idea of the preferred habitat theory is that bond markets have several “preferred habitats”, or different motives for holding bonds. Shorter-term bonds are a better choice for some investors, while others, such as an insurance company or retirement savings fund, prefer longer-term bonds (Sundaresan, 2002). As a result, there should be no tendency for term premiums to be an increasing function of maturity, as stated by the liquidity-preference theory.

However, according to Campbell (1995), for example, several studies have confirmed that the term premia is time-varying. This theory also contradicts the empirical findings concerning the formation of the term structure of interest rates. As a result, more advanced theories have been developed in order to give a better description of the TS.

2.2 Yield models

Yield models are the early formulation of the TS. One of the core elements of these models is that they include factor(s) that follow a stochastic process, usually the Brownian motion. Discrete-time and continuous-time frameworks have both been used, which, according to Cochrane (2001), is only a matter of convenience. Both frameworks have their strengths and weaknesses; this comment is also valid for MTS models. Most of these yield models use the arbitrage-free framework. In the yield models, bond yields are usually assumed to be in the logarithmic form and affine.

Not including macroeconomic variables, two of the best known models of this kind are probably those of Vasicek (1977) and Cox et al. (1985) (referred to henceforth as CIR). The Vasicek model presents a single-factor model using instantaneous default-free interest rate as the factor (Vasicek, 1977). This characterisation means that a Gaussian process[13] is all the information needed to construct this TS model. The process in question includes a drift element, which is a function of the instantaneous interest rate; this drift causes the expected value to differ from zero but still allows the instantaneous interest rate to be negative, which contradicts the definition of interest rates.

The Vasicek and CIR models are quite close to an autoregressive process (AR), where the current value, say an interest rate, is a function of past observed interest rate values. A fixed term is often added to the AR process to ensure that the expected value is nonzero. In both the Vasicek[14] and CIR models, the framework is stochastic. This means that the fixed value in a hypothetical AR process is called the drift element; that is, the "average" instantaneous short rate, which assures a mean reversion[15] property for the stochastic differential equation (SDE). As the process includes stochastic element, the instantaneous interest rate is called SDE instead of AR. In most models, including these two, the actual process that defines the stochastic nature of the model follows Brownian motion,[16] which is also called the diffusion of the SDE. After the specification of the diffusion process, that is, the stochastic part of the SDE, the arbitrage-free framework is implemented in order to gain the market price of risk (Vasicek, 1977). Then the actual bond prices can be obtained and, finally, also the actual term structure of interest rate in the Vasicek model. In practice, this follows the first order linear autoregressive process AR(1) (Vasicek, 1977).

[13] Quite often, as in this and many of the following cases, the Gaussian process is used for forecasting purposes and, as such, it is assumed that the dependent variable or its error terms are multivariate normal (Lütkepohl, 1991).
[14] The Vasicek model presented as stochastic differential equation is $dr = (\alpha + \beta r)dt + \sigma dZ$ (Chan et al. 1992).
[15] This ensures that the process varies around a certain mean so that the value does not explode to extremes such as infinity.
[16] In this context, the Brownian motion used to define the stochastic properties of this process, such as the mean, variance and joint probability distribution of the model.

The CIR model is a single-factor model,[17] as is the Vasicek model. In the CIR model, the explanatory variable is the instantaneous interest rate. The CIR and Vasicek models and six other models have been compactly presented by Chan et al. (1992). In this article, a nested model has been introduced into which all eight TS models have been computed using varying parameter values. In this framework, the markets are assumed to be free of arbitrage as was originally assumed in all of the eight considered models. The nested model shown in Chan et al. (1992) is $dr = (\alpha + \beta r)dt + \sigma r^{\gamma} dZ$, which is a SDE for the instantaneous interest rate r. The Stochastic nature of this differential equation is due to the σdZ term, where dZ is Brownian motion and σ is its fixed instantaneous standard deviation; i.e., the higher σ, the more randomness occurs in the process, which in this case is the instantaneous interest rate. Parameters α and β are constants in Vasicek as well as in the CIR model. The so-called instantaneous drift, $(\alpha + \beta r)$, ensures that the process varies around its long-term mean, γ, and is thus stationary in the long-run (Vasicek, 1977). In the Chan et al. (1992) nested model, only the stochastic part σdZ is the same for all eight TS models, while other parameters, such as γ, vary. The r^{γ} term measures the effect of instantaneous interest rate r on the level of the random component dZ. In the Vasicek model, the parameter γ is zero, which means that the level of the interest rate would have no affect on the magnitude of the σdZ. This assumption contradicts the common perception that a higher interest leads to higher standard deviation.

The outcome of Chan et al.'s (1992) empirical testing via general method of moments is that models that give high weight to the conditional volatility perform better than the alternatives. In the model framework presented earlier, the parameter γ is then required to be one[18] or $3/2$. With these specifications, four of the estimated models outperformed the other four models in fitting the model to the short-term Treasury bill data.

The Vasicek and the CIR model are special cases of the so-called affine class of TS models. There are a huge number of other TS models, which can be presented in the affine framework. This affine framework means that bond (log) yields are linear or exponentially linear

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[17] In the original paper, "A theory of the term structure of interest rate", Cox et al. (1981) did present an example of a two-factor model. Generally, however, the CIR model refers to the single-factor model.

[18] These two coefficient values were included in five of the examined models, while the value of parameter γ was lower for the other compared models.

functions of the state variables. Examples of this can be found from Duffie & Kan (1996); in other words, these models are affine in yields.

During the 1990s, these yield models evolved to multifactoral models and latent factors were also included. These were named slope, curvature and level. According to their correlation, if a latent variable had the highest correlation with the level feature of the yield curve, it was named the level, and so forth. It took about a decade before these latent factors were interpreted as macroeconomic variables in the 21st century.

A full mathematical representation of the previous and more advanced TS models led to quite cumbersome and extended presentation of all the mathematical finesses and assumptions of these models, which does not help to clarify the importance of macro models in the determination of the TS. The next sub-section looks at some insights into no-arbitrage assumption, which has become widely used in MTS models.

2.3 No-arbitrage as part of the TS modelling

It has almost become a rule to implement the no-arbitrage assumption, especially after the work of Ang & Piazzesi (2002). This influential academic article showed that the assumption of no-arbitrage improves empirical fit to their model. The other way to describe having implemented the no-arbitrage assumption is to say that the so-called cross-equation restrictions have been used.

The use of cross-equation restriction has many advantages. Firstly, it ensures that the yield dynamics are consistent; that is, no-arbitrage. Secondly, it allows the risk premia to be separated from expectations. A good example of this is that the expected returns on long maturity bonds, which are time-varying, are higher than they are on short bonds. The cross-equation restrictions make it possible to model the time variability and the expected return to the risk premia. Thirdly, the cross-equation restrictions improve the efficiency of estimated parameters, which are quite often numerous. The fourth important advantage is that this restriction can also provide very good proxies of possible missing bond yields (Piazzesi, 2010).

3 LINKAGES BETWEEN THE TERM STRUCTURE OF INTEREST RATES AND MACROECONOMICS

Macroeconomic variables have been added to TS modelling as the yield of a bond and possibly latent factors turn out to be insufficient for determining the TS. It would be quite naïve to think that the only explanatory variable needed to explain the yield curve would be its own yield. It has also been shown empirically that TS models that use only one state variable, usually the lagged interest rate, deliver very poor proxies of the TS (Cochrane, 2001). It should also be noted that if there is a variable that forecasts future TS, it should be taken as a state variable and the bond prices should reveal this (Cochrane, 2001). If only one variable, such as the bond yields, would determine the future TS, then bond yields should override any other macroeconomic state variables as the explanatory variable of the TS (Cochrane, 2001). As this is not the case when macroeconomic variables are concerned, there is a clear empirical and theoretical reason to believe that the macroeconomic state variables are part of the TS modelling and, as such, deliver more accurate forecasts of the TS than the one-factor models (Cochrane, 2001). With latent factors, this becomes somewhat more interesting as the explanatory power increases but it still does not give any explanation, either economic or otherwise, about what drives the changes in the TS.

3.1 Wicksell's interest rate policy rule

For central banks, the basic purpose of interest rate policy is to dampen economic fluctuation. Wicksell (1898) proposed that, in order to dampen the economic fluctuation, one should lower interest rate when prices fall and the opposite when prices rise. The idea was probably driven originally by the idea that the only way to develop a "rational monetary system" would be by abandoning the gold standard. He truthfully believed that the price changes were not a result of changes in the gold supply[19] but a result of policy interest rate governed by a national central bank and real disturbances that have an effect on the natural rate of interest. Mathematically, the Wickselian theory can be written as

$$i_t = \bar{\imath} + \phi p_t; \quad \Delta i_t = \phi \pi_t.$$

Here, $\bar{\imath}$ is the current interest rate and p_t is the log (originally monotonic) price index, while ϕ is a positive response coefficient for the price index. This can also be transformed to

[19] It was widely assumed in the beginning of the 20th century that the deviation in prices, i.e., inflation, was due to changes in gold supply (Woodford, 2003).

explain the change in interest rate, Δi_t, which should equal the aforementioned coefficient multiplied by the change in price level; i.e., $\pi_t \equiv \Delta p_t$. This simple rule has not been used in the MTS models but can be seen as a starting point for the more rigorous models for determining policy interest rate, such as the Taylor rule, which will be presented next (Woodford, 2003).

3.2 Taylor's interest rate policy rule

Probably the most widely used part of macro-modelling in the context of TS is the Taylor rule. In the 1990s, John B. Taylor came up with the idea that central banks could adjust their short-term interest rate in reaction to observed deviations of inflation and output from their targets, instead of trying to control the supply of money.

The European Central bank and the Bank of England can be seen as being independent enough to be committed to a fixed inflation target rate. Therefore, the Taylor rule is the theory to choose, as large parts of the data used in the models considered here have been based on European or US data. Although the Fed has not been single-mindedly committed to capping inflation, the Taylor rule is flexible enough to overcome these mild differences in the creation of monetary policy. Taylor believed that by adjusting short-term interest rates in reaction to observed deviations of inflation and output from their target values, a central bank could alleviate the negative effects of business cycles. This adjustment of short-term interest rates could be done via the nominal policy interest rate, *i*, according to the Taylor rule (SØrensen & Whitta-Jacobsen, 2005):

$$i = \bar{r} + \pi + h(\pi - \pi^*) + b(y - \bar{y}), \;\; h > 0, \;\; b > 0.$$

Here, the bars on top of a variable denote long-run equilibrium values. Inflation[20] has been denoted by π, while r is the real interest rate and the target inflation rate has been denoted by π^*. The GDP growth rate is in logarithmic form and is denoted by y (SØrensen & Whitta-Jacobsen, 2005). The coefficients h and b are positive (SØrensen & Whitta-Jacobsen, 2005) because, otherwise, high inflation and high GDP growth would have a negative effect on the nominal interest rate, which would strengthen economic cycles, thereby increasing uncertainty and economic distress. When policy-makers choose these two coefficients to

[20] In some modifications, the inflation in the Taylor rule equation denotes the inflation expectations.

dodge inflation or avoid output instability, there is no reason to assume that these coefficients would not be positive (SØrensen & Whitta-Jacobsen, 2005).

One rule of thumb that John B. Taylor (1993) presented was that the coefficients *h* and *b* in the previous equation should both be close to 0.5, because this figure would imply quite tight monetary policy. Several acknowledged studies have suggested that the coefficients *h* and *b* are positive, differing in sizes according to the central bank's preferences between low inflation and higher but more cyclical economic growth (SØrensen & Whitta-Jacobsen, 2005). There are also many other theories of policy rate determination, one of which is referred to as Neo-Wicksellian Monetary theory and is explained in detail by Woodford (2003), among others. However, because this model has not been used in the MTS models presented later on, it will not be presented here. A short presentation of a model that provides more insight to the interest rate and macroeconomic interplay is presented next.

3.3 Linkage between interest rate and macroeconomic variables

One of the more recent studies that uses the Taylor rule as a part of the TS determination is Gürkaynak et al. (2003). In this study, the widely accepted assumption of the long-run properties of economy are time-invariant and perfectly known by all economic agents. This idea has been subject to suspicion, for example in Gürkaynal et al. (2003), the empirical findings of which support the hypothesis that long-run forward interest rates often react significantly to surprises in macroeconomic data realeases and monetary policy announcements. The commonly used assumption in the macroeconomic models is that this news should only have transitory effects in the long run.

Gürkaynak et al. (2003) demonstrated that there is most probably a link between the long-term forward interest rates and the macroeconomic variables. The macro model uses three different equations to explain inflation, output gap and the short-term nominal interest rate, similar to the New Keynesian macro models. One part of the estimation is based on US news releases and the surprise component is measured by the difference of an actual figure and a published median market forecast reported by the Money Market Services. These news are related to 39 different macroeconomic statistics and also used regression analysis to study the effect of a monetary policy surprise on the forward interest rate. Both of these studies were based on US treasury forward rate data from January 1990 to December 2002.

Gürkaynak et al. (2003) concluded that many of the macroeconomic news[21] "shocks", such as retail sales, unemployment and core consumer price inflation, had long-lasting effects on forward interest rates. Among others, this study supported the fact that macroeconomic variables have and should have an effect on the TS formation.

Chapter 4 presents the so-called state-space model (SSM). This model specification is also known as the unobserved component model and has established its place in the literature as the most used framework in the resent MTS models. The idea is to give the fundamental reasons why the SSM is so widely used in the TS modelling, which has been enhanced by showing some basic properties and examples.

4 THE BASIC IDEAS OF LINEAR STATE SPACE MODELS

The aim of this section is to represent the main properties of the SSM procedures. Many of the MTS have been presented in SSM form as this is the most convenient way of representing these models. SSM can be used in most time series models and it allows a structural representation of the model. There are usually several different components that constitute the time series model, including trend, seasonal, cyclical and calendar variation with the obvious explanatory variables and possible interventions[22] (Durbin & Koopman, 2001). These components are modelled separately in the SSM framework, which gives the modeller the freedom to specify these components (Durbin & Koopman, 2001). The alternative ARIMA (autoregressive integrated moving average model), and several variations thereof, base the model construction purely on data without prior valuation of the system that generated the outcome (Durbin & Koopman, 2001). In addition, the ARIMA models and the like are homogenous in terms of their construction, which means that they are based on the assumptions that the differenced series is stationary (Durbin & Koopman, 2001). This is not the case in the SSM, which are quite flexible in this sense (Durbin & Koopman, 2001). This model specification allows known changes in the structure of the system over time (Durbin & Koopman, 2001). The next section will offer some initial jargon and general ideas, as well as some examples of SSM.

[21] There are a total of 39 macroeconomic indicators, 13 of which were chosen. These 13 showed the largest effect on the one-year forward rate. The results were almost the same if all the 39 macroeconomic indicators from Money Market Services would have been included (Gürkaynak et al. 2003).

[22] The intervention is a type of dummy variable that has a value of zero or one, which may depend, for example, on some event that either occurs (1) or does not (0). Further information on this issue can be found, for example, from Chatfield (2004).

4.1 The general form of SSM

The state space approach is used, especially by engineers, to represent time series because it is more flexible and capable than the better known ARIMA (autoregressive integrated moving average model) (Durbin & Koopman, 2001). This approach has gained momentum over time among other sciences, such as economics, due to its applicability for forecasting short-term phenomena. It has also been proven to be much easier to compute trend models and the like in the state-space form than in the ARIMA framework, for example (Chatfield, 2004).

Typically, a model is constructed in order to give a good fit to the data using the underlying data. It is also usually hoped that the model would be able to forecast future phenomena. As the fit to data is always only a proxy, a noise element with suitable conditions is usually added so that the shortcomings of the model would be less severe. Written out in word form, this idea enhances the idea of the state-space framework: Observation (dependent variable) = Signal (independent variable) + Noise (error term) (Chatfield, 2004). This equation is known as observation or measurement equation in the state space jargon and is given by:

$$(1)\ \ X_t = \boldsymbol{h}_t^T \boldsymbol{\theta}_t + n_t,$$

where the bold text refers to the column vector (later on also to matrices) and T to transposes (Chatfield, 2004). X_t is the observation, $\boldsymbol{h}_t$ (m×1) a known vector, $\boldsymbol{\theta}_t$ (m×1) is the state vector and n_t is the observation error (Chatfield, 2004).

The fundamental structure of SSM is that the signal part (state vector) constitutes of a linear combination of variables, known as state variables, which constitute the state vector (Chatfield, 2004). At time t, this state vector defines the state of the system (Chatfield, 2004). The idea of SSM is therefore to infer the relevant properties of state variables $\boldsymbol{\theta}_t$, which in turn are based on the knowledge of observations X_t (Durbin & Koopman, 2001). The state vectors include model parameters of some sort, which are usually unobservable, such as a function (Chatfield, 2004). As the state vector is unobservable, some additional assumptions need to be made in order to obtain a solution. The usual assumption for overcoming this problem is that the changes in the state vector are known ex-ante. This feature is given by a transition equation, which is also known as the state or system equation. As time goes by, the state equation updates the changes in the state vector and is given by:

$$(2)\ \boldsymbol{\theta}_t = \boldsymbol{G}_t\boldsymbol{\theta}_{t-1} + \boldsymbol{w}_t,$$

$\boldsymbol{G}_t$ denotes the (m×m) known matrix and $\boldsymbol{w}_t$ denotes a (m×1) vector of deviations, i.e., $\boldsymbol{w}_t^T = (w_{1,t}, w_{2,t}, .., w_{m,t})$ (Chatfield, 2004). Equations (1) and (2) constitute the general form of the state-space model with one variable, that is, the univariate version.

4.2 The local level SSM

An illustrative example of SSM is the local level model, which constitutes of only two scalars and one state variable. The observation equation is $X_t = \mu_t + n_t$ and the unobservable state variable, μ_t, is known as the local level (scalar instead of vector), which follows the random walk process, $\mu_t = \mu_{t-1} + w_t$ (Chatfield, 2004). Referring to the pervious equation (1) and (2), the $\boldsymbol{h}_t^T$ and $\boldsymbol{G}_t$ are left out as these are not vectors in the local level version. The error terms n_t and w_t are assumed to be independently and normally distributed with zero means and variances of σ_n^2 and σ_w^2, respectively (Chatfield, 2004).

4.3 The linear growth SSM

The more complicated models include more elements than the local level model or the general form of SSM, such as local trends and/or seasonal components. The linear growth model includes the local trend part as the basic structural form. This new feature is constructed by adding an equation to the general SSM. A system of the next three equations is needed for the specification of the linear growth model (Chatfield, 2004):

$$X_t = \mu_t + n_t; \quad \mu_t = \mu_{t-1} + \beta_{t-1} + w_{1,t}; \quad \beta_t = \beta_{t-1} + w_{2,t}.$$

As above, X_t denotes the observation equation. The two other equations are transition equations; in other words, the process of state vector $\boldsymbol{\theta}_t^T = (\mu_t, \beta_t)$ changes over time. The local level presented in the last section is still denoted by μ_t, while the additional term, β_t, determines the local trend (slope) or, alternatively, the growth of the model (Chatfield, 2004). This set of equations can easily be transformed to fit the general form.[23]

For most MTS models, the structure is comprised firstly by assuming that the macroeconomic model to be given – for example, equations for inflation rate and output gap – is based on some theoretically valid composition for a given task and, secondly, by formulating the

[23] Appendix 1 shows a full computation of this procedure.

nominal short-term interest rate, which is governed by the central bank. This set of equations is then written in the state-space form that is also known as the general form, the Markovian presentation or the canonical form. The reason for this procedure is to make the actual solution (the policy interest rate), the third phase, easier to compute by using quite sophisticated methods such as Schur decomposition.[24] Finally, the actual term structure is linked to the model by incorporating the no-arbitrage condition via stochastic discount factor and the dynamics of the market price of risk. These components make it possible to compute the continuously compounded zero-coupon bond yields or future short rates.

4.4 The Kalman filter

The state vector is usually unobserved and an estimate of this vector can be produced using the so-called Kalman Filter (KF). The outcome of this estimation is a set of equations that determine the evolution of the state vector in time, including the updating procedure of the state variables. This updating procedure has two stages, the prediction stage and the updating stage. Without going into great detail,[25] there are significant advantages to this procedure, which makes the actual computations easier in practice (Chatfield, 2004).

The first advantage is that all the “memory” needed to construct the updating procedure is short, basically comprising the previous estimate and the latest observation (Chatfield, 2004). The second point is that if the underlying model, denoted above by X_t, is constant, it converges quickly. Alternatively, if the underlying model evolves through time, it follows the movement of the system (Chatfield, 2004); that is, the “memory” quickly updates the information. There are several different variations of this KF, some of which are more complicated than others (Chatfield, 2004).

4.5 Factor model example in the state space framework

Before moving on to the MTS models, an example will clarify the relationship between the TS, yields and the state space framework. This example is based on a working paper by Diebold et al. (2006), the example model of which was based on the work by Nelson & Siegel (1987).

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[24] This is a matrix decomposition that is used to solve a SSM model given suitable transition and observation equations.

[25] For those interested in the more technical part, Chatfield (2004) gives a proper and quite easy representation of the Kalman filter and many other issues related to SSM.

The starting point is to model the yields of various maturities in a compact form. As usual, $y_t(\tau)$ denotes yields to maturity τ at time *t*. The following equation was given by Nelson and Siegel (1987) (the parameter notation was altered by Diebold et al. (2006) as used here):

$$(3)\ \ y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + C_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right),$$

L_t, S_t, and C_t are the parameters for level, slope, and curvature, respectively, and λ is one of the estimated parameters. Assume now that the dynamic process of L_t, S_t, and C_t evolve in time according to the AR(1) process; that is, the present values of the latent variables L_t, S_t, and C_t are, at least to some extent, correlated with their previously observed value.[26] With these specifics, a state space system of these variables can be formed as follows. The measurement equation (observation equation) that relates the set of *N* yields to the latent (unobservable) variables is given by Diebold et al. (2006):

$$(4)\ \ \begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\tau_1\lambda}}{\tau_1\lambda} & \frac{1-e^{-\tau_1\lambda}}{\tau_1\lambda} - e^{-\tau_1\lambda} \\ 1 & \frac{1-e^{-\tau_2\lambda}}{\tau_2\lambda} & \frac{1-e^{-\tau_2\lambda}}{\tau_2\lambda} - e^{-\tau_2\lambda} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\tau_N\lambda}}{\tau_N\lambda} & \frac{1-e^{-\tau_N\lambda}}{\tau_N\lambda} - e^{-\tau_N\lambda} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix}.$$

The transition equation (state equation), which is also part of the state space form, governs the dynamics of the state vector $(L_t, S_t, C_t)^T$ (*T* denotes transposes) (Diebold et al., 2006):

$$(5)\ \ \begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix}.$$

In equation (5), μ_L, μ_S, μ_C constitutes the mean state vector and all nine of the *a* elements are free parameters to be estimated. Observation errors are denoted by ε_t for each maturity, while η_t denotes deviations for each state t=1,...,T (Diebold et al., 2006).

[26] The AR(1) process is of the first order and, therefore, the correlation can be present only for the last observation.

State-space form is usually presented in a more compact form:

$$\boldsymbol{y_t} = \boldsymbol{\Lambda f_t} + \boldsymbol{\varepsilon_t} \ and \ (\boldsymbol{f_t} - \boldsymbol{\mu}) = \boldsymbol{A}(\boldsymbol{f_{t-1}} - \boldsymbol{\mu}) + \boldsymbol{\eta_t}.$$

Here, equations (4) and (5) are presented as a state-space system with appropriate dimensions. In order to implement the Kalman Filter, a few restrictions must be added, as follows (Diebold et al., 2006):

$$\begin{pmatrix} \boldsymbol{\eta_t} \\ \boldsymbol{\varepsilon_t} \end{pmatrix} \sim WN \left[\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H} \end{pmatrix} \right] \ and \ E(\boldsymbol{f_0} \boldsymbol{\eta_t^T}) = 0, E(\boldsymbol{f_0} \boldsymbol{\varepsilon_t^T}) = 0.$$

The first of these equations means that deviation $\boldsymbol{\eta_t}$ has zero mean and variance determined by variance-covariance matrix $\boldsymbol{Q}$, which is non-diagonal; in other words, a part of the elements that are not in the diagonal are left to be non-zero so the three state variables might be correlated with each other, and will be if the non-diagonals are not zero. The same reasoning applies to the observation errors except that these errors are not correlated with each other (variance-covariance matrix $\boldsymbol{H}$ is a diagonal matrix). The second part of these equations ensures that $\boldsymbol{\eta_t}$ and $\boldsymbol{\varepsilon_t}$ are orthogonal to initial state $\boldsymbol{f_0}$; that is, there is no statistical relationship between the initial value of the state variable and the error terms ($\boldsymbol{\eta_t}$ and $\boldsymbol{\varepsilon_t}$). *WN* is a shortening of the wrapped normal distribution, which means that the normal probability density function is wrapped around a unit circle instead of values on the horizontal-axis and density on the vertical-axis. The next step would be to use the Kalman filter in order to obtain optimal yield and error predictions, after which a version of the maximum likelihood estimation could be used to obtain the missing parameter estimates[27] (Diebold et al., 2006).

The next section presents a modified version of the previous model by Diebold et al. (2006). This version also uses the VAR factor framework but utilises the macro variables and permits two-way interactions from yields to macro variables and vice versa. This feature is desirable

[27] In this case, the free parameter space would be quite large (nine parameters from matrix A and three from the mean state vector $\boldsymbol{\mu}$, while matrix $\boldsymbol{Q}$ would have six free parameters and the number of free parameters in matrix $\boldsymbol{H}$ would depend on the amount of different maturity yields in the estimation). Despite this, it would still be manageable for the Kalman filter and maximum likelihood estimation (Diebold et al., 2006).

as some studies have shown that TS can and probably does work as an indicator for other economic variables, such as consumption and investment (Estrella & Hardouvelis, 1991).

5 TERM STRUCTURE MODELLING IN THE MACROECONOMIC FRAMEWORK

The main difference between various MTS models is in the way the macroeconomy has been modelled and, because of this, the macro models lie at the heart of this thesis. Also, the specification of the market price of risk differs between several studies, although it seems that the time-varying market price of risk has gained popularity since Duffee (2002) and Hödahl et al. (2006), which support its use from a theoretical perspective. There are other similarities between the newest MTS models; the absence of arbitrage, for example, is almost a rule, as is the affines of yields. The most illustrative categorisation of the MTS models is probably gained by differentiating the models from each other through the characterisation of the macroeconomy. An illustrative table of different MTS models below emphasises the differences and similarities between different MTS studies.

Table 1

	ARBITRAGE (Y/N)	MARKET PRICE OF RISK	AFFINE IN YIELDS (Y/N)	MACRO FRAMEWORK
VAR	N	Time-varying	Y	Variables
HTV	N	Time-varying	Y	New Keynesian
VAR FACTOR	Y	Fixed	Y	Variables
LEMKE	N	Time-varying	Y	Structural model
DSGE	N	Time-varying	Y	DSGE

On the left hand side is the abbreviation of the MTS models that will be introduced. The next column indicates whether arbitrage is allowed to occur (Y) or not (N) in the given model.

For many MTS models, the structure is comprised by giving the macroeconomic model, which could be done, for example, by formulating the inflation rate and output gap and, lastly, by defining the nominal short-term interest rate, which is governed by the central bank. This set of equations is then written in the state-space form. Finally, the actual term structure is linked to the model by incorporating the no-arbitrage condition via the stochastic discount factor and the dynamics of the market price of risk. With these components, the continuously compounded zero-coupon bond yields, or the future short rate, can be computed.

Another distinctive feature of these MTS models is that the structural models are based on the New Keynesian macro models, while the DSGE models are based on a mixture of New Classiscal and New Keynesian synthesis. The DSGE model of macroeconomic dynamics is used at least by the European Central Bank (ECB) and fairly deserves representation as the newest part of the MTS modelling, although ECB uses it to monitor macroeconomic activities.

The basic framework for MTS modelling was provided by Duffie & Kan (1996), in which they introduced a so-called standard affine term structure framework based on latent variables (Hördahl et. al, 2006). The latent variables in their work reflect the properties of the TS yield curve, namely the slope, curvature and level. In other words, these variables had no economic interpretation but can be seen as the starting point for further development introduced by Ang & Piazzesi (2003), which used the VAR framework in order to nest the macro variables to the TS. This progress was followed by several other studies, most of which were implemented with the support of central banks in Europe, the UK and the US.

The similarity of the structural and VAR models lies in the fact that they both depend on the New Keynesian framework and the TS modelling has been carried out by adding flexible features to the representative models (De Graeve et. al, 2009). These flexible attributes include time-varying parameters, time-varying variances of structural shocks and flexible pricing kernels (De Graeve et al., 2009). Examples of these flexibilities include the HTV model, in which the additional shocks have been implemented in order to improve the fit of the model to the data, and the Ang & Piazzesi (2003) model, for which latent variables have provided the necessary flexibility (De Graeve et al., 2009).

MTS modelling has been a popular and dynamic line of academic research recently, which means that the scientific atmosphere has been constantly evolving. This thesis separates the MTS into three categories: Structural, Vector autoregression (VAR) and DSGE models. The MTS models that are based on the dynamic stochastic general equilibrium (DSGE) represent the newest form of macro modelling and will therefore be presented last, followed by some concluding thoughts about the models. This is followed by a presentation of the VAR MTS models, including the Ang & Piazzesi (2003) model, the structural framework and, finally, the presentation of the DSGE MTS model.

5.1 VAR model

Models that use the VAR presentation are presented in this section, with particular focus on those ones that use macroeconomic variables to explain TS. The earliest study using the VAR systems of yields was Sargent (1979), who used it under the null of the expectation hypothesis. It took many years before the macro variables were included in the VAR framework. Estrella & Mishkin (1997) and Evans & Marshall (1998) were among the first to include the macro variables in TS modelling. Estrella & Mishkin (1997) examined the effects that monetary policy, real activity and inflation have on TS. Evans & Marshall (1997), on the other hand, examined how exogenous impulses on monetary policy affect the yield curve. These two studies included some inconsistent assumptions. Neither included cross-equation restrictions, which allows arbitrage, and both used unrestricted VAR, which means that they refer only to the movements of yields that are included in the model (i.e., there is no usage for future forecasts). The third shortcoming is that only observed variables can be used; there is no room for the latent variables that have become essential part of the MTS models.

Work by Ang & Piazzesi (2003) solved some of these shortcomings. Their work was greatly affected by Duffie & Kan (1996), in which the state variables were latent. Ang & Piazzesi (2003) changed this setup by adding observable macroeconomic aggregates, inflation and real activity measured in various ways. The main conclusion from Ang and Piazzesi's study was that the no-arbitrage condition and the addition of macro variables increase the explanation power of a VAR model. This was a remarkable discovery and Ang & Piazzesi's paper remains one of the most cited in the MTS literature. However, there has been increasing criticism of the no-arbitrage hypothesis, for example from Diebold et al. (2006).

Ang & Piazzesi (2003) also observed that from June 1952 to December 2000 the distribution of yields of differing maturities did not follow a normal distribution. However, they also pointed out that their aim was to study the joint dynamics of yields and macro variables, not the structure or other features of yields or macro variables separately. On this basis, their Gaussian model is adequate for estimation despite the skewed distribution of yields.

An interesting feature of Ang & Piazzesi (2003) is that macro variables (inflation and real activity) were estimated using several different measures. Inflation was measured via CPI (consumer price index), PPI (producer price index) of finished goods and PCOM (spot market commodity prices). The real activity, in turn, was measured by HELP (help wanted advertising in newspapers), UE (unemployment rate), EMPLOY (the growth rate of employment, and IP (industrial productivity). PCOM and HELP have traditionally been used as leading indicators of inflation and real activity, respectively. An example of PCOM and

HELP used as indicators is that an increasing economic activity has quite often been present when both PCOM and HELP figures are higher than in the previous period. In other words, increasing values of PCOM and HELP indicates heightened economic activity.

Because the measurement space to use all these attributes would have been quite large, Ang & Piazzesi (2003) used the principal component analysis[28] to reduce the dimensionality. This was done by incorporating all the inflation and real activity-related variables into two vectors, namely inflation Z_t^1 and real activity Z_t^2. These vectors (dimensions of three and four, respectively) were represented as:

$$\boldsymbol{Z}_t^i = \boldsymbol{C}\boldsymbol{f}_t^{o,i} + \boldsymbol{\varepsilon}_t^i.$$

Here, all the terms are presented as vectors of appropriate dimensions (either 3×1 or 4×1) (Ang & Piazzesi, 2003). The error term $\boldsymbol{\varepsilon}_t^i$ has an expected value of zero and its variance is a diagonal variance-covariance matrix with only diagonal entries (i.e., error terms are uncorrelated) (Ang & Piazzesi, 2003). The coefficient matrix $\boldsymbol{C}$ is the so-called factor loading, which can be seen as a weight coefficient for various inflation and real activity variables. Using the principal component analysis makes it possible to extract the macro factor $\boldsymbol{f}_t^{o,i}$ (here *o* denotes observed) and it exhibits zero mean and unit variance. Although this is a convenient and widely used manipulation of data, it also comes with some shortcomings as it is an aggregation of the used data. As Ang & Piazzesi (2003) pointed out, the outcome of the real activity factor analysis shows quite a different estimate than if treated separately. For example, HELP shows a high correlation with a one-month yield, while EMPLOY and IP contain the greatest loadings (weights) in the real activity factor.

Three arguments can be used to justify the VAR(12) presentation of the extracted macro variables used in Ang & Piazzesi (2003). The first of these is that 12 lags represent a year, as the observation interval is one month. The second is the presence of so-called habit formation, that is, a consumer is expected to consume roughly the same as in the previous period or, if the income increases, part of the increased consumption potential is postponed to the next

[28] This technique reduces the number of variables without losing vital information from the covariance matrix (Campbell et al., 1997). A more profound representation of principal component analysis can be found, for example, in Campbell et al. (1997).

period, which is in line with the use of lags in the real activity modelling. Thirdly, the lags in the inflation model are well justified as wages and prices adjust slowly to shocks.

Here the bivariate VAR(12) process[29] of inflation and real activity, factors has been given as:

$$(6)\ \boldsymbol{f}_t^o = \boldsymbol{\rho}_1 \boldsymbol{f}_{t-1}^o + \cdots + \boldsymbol{\rho}_{12} \boldsymbol{f}_{t-12}^o + \boldsymbol{\Omega} \boldsymbol{u}_t^o.$$

In this equation inflation and real activity factors are given by $\boldsymbol{f}_t^o = \left(\boldsymbol{f}_t^{o,1}\ \boldsymbol{f}_t^{o,2}\right)$, and $\boldsymbol{\Omega}$ is a coefficient matrix 2×2 (Ang & Piazzesi, 2003). The vector of errors $\boldsymbol{u}_t^o$ is independently, identically and normally distributed with zero mean and unit variance in mathematical notation $\boldsymbol{u}_t^o \sim IID\ N(0, \boldsymbol{I})$[30] (Ang & Piazzesi, 2003).

The affine nature of TS models is based on a short rate equation and on the assumption on risk premia (Duffie & Kan, 1996). Also, there are similarities between the Taylor rule and affine term structure models. The difference between the Taylor rule and affine TS models is that the explanatory variables in the Taylor rule are observable, while at least some of the variables in the affine term structure models are constructed as unobservable latent factors as in Ang & Piazzesi (2003). The next three equations for the short interest rate by Ang & Piazzesi (2003) emphasise this point:

$$(7)\ r_t = a_0 + \boldsymbol{a}_1^T \boldsymbol{f}_t^o + \boldsymbol{v}_t;\quad r_t = b_0 + b_1^T \boldsymbol{X}_t^o + \boldsymbol{v}_t;\quad r_t = c_0 + \boldsymbol{c}_1^T \boldsymbol{X}_t^u.$$

All the bolded terms here are row vectors and T denotes transposes as before. The first equation is the Taylor representation,[31] in which $\boldsymbol{v}_t$ presents a policy shock as proposed by Christiano et al. (1996). The second equation is a forward-looking Taylor rule that also incorporates lags that were originally proposed by Clarida et al. (2000). The third equation presented in (7) is an affine term structure model in which the unobserved (latent) factors

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[29] The bivariate process is due to two extracted macro variables: inflation and real activity.

[30] Independency of other error terms means that no outcome has an effect on another. All error terms share the same probability distribution, in this case a normal one. $\boldsymbol{I}$ refers to diagonal matrix with ones on the diagonal and zeroes elsewhere. In this context it is the covariance matrix, where the covariance terms (off diagonal) are zeroes and the diagonal constitutes of ones.

[31] In the background and motivation, the Taylor rule included $h(\pi - \pi^*) + b(y - \bar{y})$, which has been presented here in matrix form as $\boldsymbol{a}_1^T \boldsymbol{f}_t^o$.

have been denoted by $\boldsymbol{X}_t^u$, which also follows an affine process. Given the risk-neutral pricing, that is no-arbitrage assumption, the bond prices are exponentially affine functions of $\boldsymbol{X}_t^u$, the latent factors (Ang & Piazzesi, 2003).

The aim of this VAR model is to capture the information from the macro forecasts. This can be done by adding lagged macro variables to the first equation of (7) by writing $\boldsymbol{X}_t^o = \left(f_t^{o^T} f_{t-1}^{o^T}, .., f_{t-12}^{o^T}\right)^T$. This procedure yields the following function for the short-term interest rate: $r_t = b_0 + \boldsymbol{b}_1^T \boldsymbol{X}_t^o + \boldsymbol{v}_t$ (that is, the Taylor rule with lagged observable macro variables). $\boldsymbol{X}_t^o$ captures the $\boldsymbol{f}_t^o$ term with their lags as in (6), and $\boldsymbol{b}_1$ denotes the coefficients $\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_{12}$ in (6). The third equation in (7) is the affine version of the short rate, where the short rate is an affine function of underlying latent factors $\boldsymbol{X}_t^u$. The outcome of combining the equations in (7) is:

$$(8)\ \ r_t = \delta_0 + \boldsymbol{\delta}_{11}^T \boldsymbol{X}_t^o + \boldsymbol{\delta}_{12}^T \boldsymbol{X}_t^u,$$

where the coefficient matrices $\boldsymbol{\delta}_{11}^T$ and $\boldsymbol{\delta}_{12}^T$ dimensions are 2×1 and 3×1, respectively (Ang & Piazzesi, 2003). Observable factors are $\boldsymbol{X}_t^o = \left(\boldsymbol{f}_t^{o^T}, \boldsymbol{f}_{t-1}^{o^T}, \ldots, \boldsymbol{f}_{t-11}^{o^T}\right)^T$ while the unobserved factors $\boldsymbol{X}_t^u = f_t^u$ equal the contemporaneous latent yield factors: slope, curvature and level (Ang & Piazzesi, 2003).

The next step is to construct a model that incorporates equation (6), the short-term interest rate and the market price of risk that varies in time. There are two observable macro variables, $\boldsymbol{f}_t^o$, and three unobservable $\boldsymbol{f}_t^u$ ones. Let the vector $\boldsymbol{F}_t = \left(\boldsymbol{f}_t^{o^T}, \boldsymbol{f}_t^{u^T}\right)^T$ follow a Gaussian VAR (12) process, given by:

$$\boldsymbol{F}_t = \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \boldsymbol{F}_{t-1} + \cdots + \boldsymbol{\Phi}_{12} \boldsymbol{F}_{t-12} + \boldsymbol{\theta} \boldsymbol{u}_t.$$

As this is a Gaussian VAR, the disturbance vector $\boldsymbol{u}_t$ must be independent and identically distributed (IID) with zero mean and unit variance $(\boldsymbol{u}_t \sim IID\ N(\boldsymbol{0}, \boldsymbol{I}))$. The latent factors have been determined by an AR(1) process $\boldsymbol{f}_t^u = \boldsymbol{\rho} \boldsymbol{f}_{t-1}^{\boldsymbol{u}} + \boldsymbol{u}_t^u$, so the unobservable part $(\boldsymbol{\Phi}_2 \ldots \boldsymbol{\Phi}_{12})$, i.e., from *t-2* to *t-12* in the previous equation, has an outcome equal to $\boldsymbol{f}_t^u = 0$.

The summation of latent and macro variables constitutes the dimension of state vector[32] $\boldsymbol{X}_t = \left(\boldsymbol{X}_t^{o\prime}, \boldsymbol{X}_t^{u\prime}\right)^{\prime}$(Ang & Piazzesi, 2003).

This state vector is closely related to the short rate equation r_t, which is an affine function of all state variables $r_t = \delta_0 + \boldsymbol{\delta}_1^{\prime}\boldsymbol{X}_t$. The macro model is obtained when the coefficient vector $\boldsymbol{\delta_1}$ depends only on present time factor values, not on their lags. This corresponds to the Taylor rule presented in the first equation of (7). On the other hand, if the coefficient vector $\boldsymbol{\delta_1}$ would be unconstrained, then the short rate equation would also use the lagged values of macro variables. This would equal the forward-looking Taylor rule with lags, i.e. the second equation in (7) (Ang & Piazzesi, 2003).

The market price of risk $\boldsymbol{\lambda}_t$ is also an affine process: $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1\boldsymbol{X}_t$. This equation relates the price of risk to uncertainty via the so-called Radon-Nikodym derivative, which is used to convert the risk-neutral measure to an observable one (Ang & Piazzesi, 2003).

A full expression of the state space formulation would be too long to present in this context. However, the idea is that, via the state space formulation, the absence of arbitrage and the stochastic discount factor would be combined with the time-varying market price of risk. These issues ensure the existence of the risk-neutral measure, which in turn ensures that all discounted bond prices are martingales and, as such, enable the affine structure of bond yields (Shreve, 2004).

The estimation procedure was performed in Ang & Piazzesi (2003) using the maximum likelihood method. Ang and Piazzesi noted that the estimation for their model should be done in two phases (they tried estimating all the parameters at once but this resulted in non-stationary dynamics). In the first step, the macro dynamics of equation (6) and coefficients (δ_0, δ_{11}) of short-rate dynamics equation (8) were estimated using OLS regression (Ang & Piazzesi, 2003). These values were then fixed and the remaining parameters (latent variables and parameters related to the price of risk) were estimated using the maximum likelihood method.

[32] There are three latent variables: slope, level and curvature. Adding the observable variables (inflation and real activity) makes a total of five state variables.

Ang & Piazzesi (2003) did not specify the order of the three latent variables; instead they induced the level, slope and curvature of the yield curve[33] from the autocorrelations of these latent variables, as several other studies have done previously. The level transformation has been defined[34] as $(y_t^1 + y_t^{12} + y_t^{60})/3$. One of the unobserved factors has a 92 percent correlation with the level factor, so it has been chosen to correspond to the level. The slope transformation has been defined as the "spread" between the five-year yield and the one-month yield $(y_t^{60} - y_t^1)$. This has a 58 percent correlation with one of the remaining two latent variables. Finally, the curvature transformation has been defined as having a high correlation with $y_t^1 - 2y_t^{12} + y_t^{60}$. The correlation with the last latent variable and curvature was 77 percent.

In the Ang & Piazzesi (2003) model, the relative contribution of macro and latent factors can be measured via the forecast variances by constructing variance decomposition.[35] The idea is to compare the amount of influence each factor has on other factors or, in other words, the extent to which the (unconditional) variance of, say, a macro variable, can explain the one-month yield when the forecast horizon is one year. According to this analysis from Ang & Piazzesi (2003), the largest effect (85 percent) that a macro variable has is on the one-month yield with the macro lag model with unconditional variance.[36] This relation decreases as the maturity of yields increases, reaching the minimum at five-year maturity with unconditional variance in the case of the macro lag model.

The Taylor rule is, in this case, constructed in such a way that the inflation and real activity contribute most of the variation and the latent factors only explain the residuals, that is, the part that is not explained by macro variables. This way of constructing the Taylor rule is probably one of the biggest reasons why the macro factors explain such a large proportion of the variance decomposition. This seems to be quite an accurate assumption given that the Ang

[33] These three latent variables do not determine the level, slope and curvature of the yield curve but they have an effect on these attributes (Litterman & Scheinkman, 1991).
[34] The first term in the brackets corresponds to the one-month yield of a zero-coupon bond at time *t*; i.e., the superscript is the remaining maturity in months and the subscript denotes the present time *t*.
[35] Further knowledge of the variance decomposition, which is also known as forecast error variance decomposition, can be found, for example, in Lütkepohl (1991), starting from page 56.
[36] The unconditional variance is in question when the forecast horizon is set to infinity (Ang & Piazzesi, 2003).

& Piazzesi (2003) model outperforms both the random walk hypothesis and the yields-only model for all forecast horizons in which the out-of-sample method is used. The only exception is the random walk hypothesis, which delivered more accurate forecasts for the three-month period.[37]

In Ang & Piazzesi (2003), the five-year yield that is forecasted five years ahead in the macro lag model has a variance that is explained up to 11 percent by macro-variables in the macro lag model. Most of the variation is caused by the most auto-correlated latent factor, the persistent level factor that accounts for 86 percent of the variation in yields. Cholesky orthogonalisation[38] was used to overcome the problem with the correlation between the inflation and real activity (Ang & Piazzesi, 2003). This procedure makes the chosen variables uncorrelated, as was originally assumed for the macro and latent factors.

Probably the most interesting part in Ang & Piazzesi (2003) is the out-of-sample forecast[39] results. The forecasts were made for the last five years of the sample period (12/1995–12/2000). In addition to the models presented before (the Yields-only Model, Macro Model and Macro Lag Model), the often used random walk (RW) model is also conducted with VAR (12), with yields only as well as with macro variables, in both VAR cases without the no-arbitrage assumption[40] (Ang & Piazzesi, 2003).

In order to compare the results between these three models, the Root Mean Squared Error, RMSE,[41] and the Mean Absolute Deviation, MAD,[42] were used (Ang & Piazzesi, 2003). This

[37] The RW hypothesis is expected to outperform the Ang and Piazzesi model as the data from macro variables is always historical and, in some cases, not totally accurate when released.
[38] The General idea of Cholesky orthogonalisation is to overcome the problem of residual cross-correlation (Hein & Truger, 2007). The first variable, e.g. CPI in inflation equation, has an effect on all other variables and the second PCOM affects others but not the previous one(s) and so forth (Hein & Truger, 2007). The shortcoming of this procedure is that the results are quite sensitive to the ordering of variables (Hein & Truger, 2007). However, Ang and Piazzesi argued that the ordering did not have a significant effect on the outcome.
[39] The main idea is to test the model with data other than that used to fit the model to the observations (Tashman, 2000).
[40] For the Yields-Only Model, the Macro Model and the Macro Lag Model, the forecast comparisons have been done by using the no-arbitrage condition (Ang & Piazzesi, 2003).
[41] RMSE is a commonly used measurement for precision, i.e., how well an estimator models the actual data as an aggregate (formally $\sqrt{E\{(Z-\theta)^2\}}$). Here, Z is the estimator in this case the actual yield and θ is the estimated (parameter) yield.

comparison shed light on the forecast ability of the models that were chosen to be compared (Ang & Piazzesi, 2003). The outcome of this experiment was expected and the addition of no-arbitrage to the models yielded statistically better results. The performance of the VAR models, which do impose the no-arbitrage condition, is close to that of RW (which does not impose the no-arbitrage condition), while the unrestricted VAR performed poorly compared to RW or to the VAR, which does impose the no-arbitrage assumption (Ang & Piazzesi, 2003).

The most intriguing result is the performance of the Macro Model and Macro Lag Model compared to other models. Firstly, the Macro Model outperformed the Macro Lag Model in forecasting, regardless of which criteria (RMSE or MAD) was used (Ang & Piazzesi, 2003). The Macro Model also outperformed the RW hypothesis in every yield forecast except the three-month yield (Ang & Piazzesi, 2003).

Although promising, these results lean significantly towards the latent variables, especially on the so-called level factor, which has a significant and, in fact, almost identical effect on the Yields-Only Model and the Macro Model. The natural next step is towards models that do not use these latent variables in their modelling. The next section introduces the model presented by Hördahl et al. in 2006, which is a great example of a MTS model that does not use latent factor(s) as explanatory variable(s). In this model, the short rate and law of motion of state variables have been obtained endogenously as in any modern macroeconomic model (Hördahl et al., 2006).

5.2 Structural MTS models

When it comes to the structural MTS models, the paper by Hördahl et al. (2006) (henceforth, HTV) is well constructed from both a theoretical and empirical perspective. The paper effectively presents the restrictions and possibilities that the structural framework delivers as it uses out-of-the sample estimation and contains extensive empirical and theoretical components.

[42] MAD measures the average absolute deviation of observations from their forecast, i.e., the size of the deviation of each observation from its mean (formally $\frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$) where n is the number of observations, x_i is the actual observation and $\bar{x}$ is the mean of all observations.

The absence of arbitrage restriction has one significant shortcoming in the TS modelling. Although bond markets can be assumed to be efficient and possible arbitrage opportunities are traded away as soon as the opportunity arises, there is still one major potential problem when it comes to the modelling of MTS. If the underlying model is mis-specified, then the restrictions placed on the model, such as the freedom of arbitrage, would most probably decrease the empirical performance of a model (Diebold et al., 2005).

Several different versions of the MTS models belong to the structural framework; therefore, three different versions will be presented in order to give an idea of how versatile this group of models are. Another supporting reasoning is the fact that most of the recent academic publications related to MTS are in fact constructed in the structural framework.

5.2.1 The HTV model (2006)

The structural framework in the HTV model means that the macroeconomic part of the modelling is strictly structural; that is, there are no AR processes that would govern the process of the macro variables, as in the VAR models. In the HTV model, only the market price of risk and target inflation rate has been modelled exogenously. This separates the HTV model from many other structural models, such as Evans (2003) and Ang & Bekaert (2004), which rely more heavily on exogenous modelling. Consequently, the HTV model can be seen as more theoretically valid than many other "competing" models, in which short-term interest rates or some other part of the TS framework has been modelled exogenously. A significant part of the empirical results are based on the restrictions laid on the market price of risk. The equation of market price of risk defines the process of term premium[43] in time (Diebold et al., 2005). This premium has been modelled as time varying, which has gained support from various empirical studies, such as Ang & Piazzesi (2003).

Before moving on to the market price of risk, the logical starting point is the model characterisation via the macroeconomic equations. In the HTV model, the equation for the inflation is given by:

[43] The expectations hypothesis assumes, quite logically, that a long-term bond held to maturity should have the same rate of return as a number of shorter-term bonds held to the same maturity. However, this does not take into account the fact that inflation and other possible changes in the economy might change the risk during the maturity, so the risk-averse investor would therefore require compensation for bearing this risk. This compensation is the term premium.

$$(9)\ \pi_t = \frac{\mu_\pi}{12}\sum_{i=1}^{12} \mathbb{E}_t\,[\pi_{t+i}] + (1-\mu_\pi)\sum_{i=1}^{3} \delta_{\pi i}\,\pi_{t-i} + \delta_x x_t + \varepsilon_t^\pi.$$

Here, π_t denotes the inflation at time t in logarithmic form, i.e., $\pi_t \equiv lnP_t - lnP_{t-12}$. The parameters to be estimated are the weight given to the expected value of the next period's inflation, $\frac{\mu_\pi}{12}$, and $(1-\mu_\pi)$, which is the weight given to all the three lags of inflation (π_{t-i}). This idea has been used by Christiano et al. (2001), in which the idea of partial price indexation to inflation has been implemented in this very same context. This means that the current price level set by firms is a function of previous prices and inflation rates. This simple rule has been used widely in the literature when it has been assumed that a firm cannot re-optimise its price forming function (Christiano et al., 2001). The disturbance term of inflation is ε_t^π and the output gap is denoted by x_t. The time periods here have been measured in months and $\delta_{\pi i}$ is the coefficient for the lagged inflation for each three lags. This coefficient has a restriction that is consistent with a form of the natural rate hypothesis, i.e., $\sum_{i=1}^{3} \delta_{\pi i} = 1$. The justification for this restriction in this context is given by Rudebush (2002) with sufficient *p*-value (0.24). The New Keynesian inflation and output gap equations by HTV (9) and (10) are also related to the work of Rudebush (2002). The elasticity of inflation to output gap is given by parameter δ_x. One conclusion is that equation (9) determines the process of inflation, which depends on inflation expectations from one month up to a year, represented in one-month intervals, and of lagged inflation up to three months. The output gap has also been assumed to have an effect on inflation. This model is consistent with the New Keynesian theory; only a few changes have been made in order to be consistent with other studies that use monthly data instead of yearly data. The equation for the output gap x_t in HTV (2006) was:

$$(10)\ x_t = \frac{\mu_x}{12}\sum_{i=1}^{12} \mathbb{E}_t\,[x_{t+i}] + (1-\mu_x)\sum_{i=1}^{3} \zeta_{xi}\,x_{t-i} - \zeta_r(r_t - \mathbb{E}_t[\pi_{t+11}]) + \varepsilon_t^x.$$

This is one variation of the New Keynesian aggregate demand function. Here the notation follows the same logic as in equation (3), except that the third part on the right-hand side denotes the real short-term interest rate. This equation of output gap can be derived from an intertemporal consumption Euler equation (Hördahl et al., 2006). The nominal short-term interest rate has been denoted by r_t at time t and $\mathbb{E}_t[\pi_{t+11}]$ is the expectation of inflation t+11 at time *t*. This form of expectation was originally introduced by Hall in 1978 as a version of the random walk hypothesis. The parameter coefficient for the real short-term interest rate is ζ_r and ζ_{xi} for the three lags of output.

In order to solve the rational expectations equilibrium,[44] an assumption must be made regarding how the monetary policy will be conducted. The central bank has been assumed to govern the short-term nominal interest r_t according to the following equation:

$$(11)\ \ r_t = (1-\rho)(\beta(\mathbb{E}_t[\pi_{t+11}] - \pi_t^*) + \gamma x_t) + \rho r_{t-1} + \eta_t.$$

This equation was originally formulated by Clarida et al. (1998), where β and ρ are parameters for the inflation gap (the gap between the expected inflation and the inflation target) and for short-term nominal interest rate smoothing,[45] respectively. The error term is given by η_t, while $\mathbb{E}_t$ denotes the expectation parameter at time *t*, as before, and γ is the parameter that corresponds to output gaps effect on r_t. The interpretation of this equation is that the nominal interest rate is a function of expected inflation at time $t + 11$, target inflation π_t^* (defined next), output gap and previous period's interest rate r_{t-1}. This is a Taylor-type rule with interest rate smoothing. As the inflation target is time-varying, it is also unobserved, as defined by:

$$(12)\ \ \pi_t^* = \phi_\pi \pi_{t-1}^* + u_{\pi,t}.$$

This equation follows the AR(1) process. In other words, the inflation target rate is determined by the lag of the inflation target and the disturbance term that is serially uncorrelated (no autocorrelation with other disturbance terms such as ε_t^x in (10)) with zero mean and a constant variance through time.

The HTV model defines the macroeconomy in a structural fashion. Equations (9)–(12) comprise the discrete time macroeconomic framework of the HTV model. The state space

[44] Under the rational expectations hypothesis, it is assumed that individuals use all the available information in order to form expectations of the future. These expectations may or may not prove correct but they will not deviate systematically from the true values.
[45] The higher ρ is, the larger the effect of last observed short-term nominal interest rate r_{t-1} on the present r_t.

form has been used in order to connect the term structure to the macroeconomic formulation. This allows a proper formulation of dependence between different variables that is more dynamic than VAR models in which the inflation and the output has to be independent of the policy rule (Hördahl et al., 2006).

The HTV model outperforms many other earlier term structure models in terms of yield forecasting (especially for longer time periods) and seems to perform quite well in the out-of-the-sample context. Four models were compared against each other, the first of which was the $A_0(3)$ model by Dai & Singleton (2000), a three-factor model. This model was chosen because Duffee (2002) showed that it is, theoretically, the most valid affine three-factor model for forecasting US yields. In the use of $A_0(3)$, a different data set was used so the results are not directly comparable. The second model, the Ang & Piazzesi (2003) model, was just introduced, in which the interest rate responds to the current inflation and output gap. Three unobserved latent variables were also included. The third model for comparison was the unrestricted VAR model, which is not structural and does not impose a no-arbitrage condition. The fourth and empirically most difficult hypothesis to beat was the random walk.

In the out-of-the-sample forecasting, the HTV model outperformed the aforementioned models in 60 percent of the cases (Hördahl et al., 2006). Out-of-the-sample estimates of the VAR model with the same variables as in the structural HTV model seemed to outperform the HTV model in the shorter horizons (Hördahl et al., 2006). However promising these results were, there are still several theoretical shortcomings in the structural framework, some of which will be discussed below.

For the HTV model, as with many others, the exchange rate was not part of the macro model, and several other variables, such as the employment rate, had also been omitted (Hördahl et al., 2006). However, allowing the number of state variables to increase substantially would create several difficulties, and this would be the case if some or all of these possible variables were included in the modelling. One of these difficulties has been mentioned in HTV, namely that the result of the huge parameter space would increase the risk of autocorrelation between the error terms. There would probably also be a discrepancy in terms of determining which variables to include and which to omit.

All in all, the HTV model performs well in term structure modelling. While there is room for development, the HTV model has brought many interesting features together, such as the endogenous modelling of the leading macroeconomic state variables to the field of TS modelling.

5.2.2 VAR factor model (2006)

This section is a continuation of section 4.5, which presented the pure yields-only model by Diebold et al. (2006). Here the level, slope, and curvature – that is, the latent variables – have been supplemented with three macro variables. As this model includes latent variables, it already is quite different to that of the HTV model. Probably the most significant difference, however, is that Diebold et al. (2006) do not impose the arbitrage freedom restriction, arguing that some bonds in the market obtain a low volatility and therefore fail to meet the requirements of the no-arbitrage framework. There are also mixed views about how much interference the no-arbitrage restriction brings to the modelling (Diebold et al., 2006). The justification for the no-arbitrage framework was given later on, however, when Christensen et al. (2007) published a working paper using the same Nelson-Siegel framework, enhanced with the no-arbitrage restriction. They concluded that implementing the no-arbitrage restrictions gave much better estimates than the one done without it. This applies when fitting the yields at a particular point of time. If, on the other hand, one seeks a fit of yields over time, the no-arbitrage condition performs quite poorly and this statement can be verified, for example, by Duffee (2002) and Brousseau (2002). As the aim is to relate the evolution of the yield curve to macro variables over time, the abandonment of the no-arbitrage condition, at least according to these backgrounds, actually seems quite justifiable. For the time being, however, there have not been any influential studies that would support this view empirically.

The notation in this section is slightly different from the one shown before. $\hat{L}_t, \hat{S}_t, \hat{C}_t$ are the level, slope and curvature, respectively, and the aim is to get a clearer view of how these three state variables interact with the macroeconomic variables. These macroeconomic variables constitute of manufacturing capacity utilisation $(\boldsymbol{CU}_t)$, the federal funds rate $(\boldsymbol{FFR_t})$ and annual price inflation $(\boldsymbol{INFL_t})$. The needed equations and restrictions are (Diebold et al., 2006):

$$(\boldsymbol{f_t} - \boldsymbol{\mu}) = \boldsymbol{A}(\boldsymbol{f_{t-1}} - \boldsymbol{\mu}) + \boldsymbol{\eta_t},$$
$$\boldsymbol{y_t} = \boldsymbol{\Lambda f_t} + \boldsymbol{\varepsilon_t},$$

$$\begin{pmatrix} \boldsymbol{\eta}_t \\ \boldsymbol{\varepsilon}_t \end{pmatrix} \sim WN \left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{Q} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H} \end{pmatrix} \right].$$

Although the empirical results do not contain estimation for the yield curve forecasting abilities, they do include the linkage between the macro factors and the yield curve, which builds the fundamental assumptions for MTS modelling, namely that macro variables have an effect on the yield curve and vice versa. The empirical results imply that using the yields-only model provides almost the same coefficients for the slope, curvature and level as the MTS model above. The differences occur in the variance decomposition used to analyse macro and yield curve interactions. It states that idiosyncratic (extraordinary) variation is present in the short-term yields but that, for longer horizons, macro factors become more influential and, at the 60-month horizon, the macro factors account for 40 percent of the variation in interest rates. According to the variance decomposition, the yield curve has a minor effect on the macro factors, while the macro factors have a more significant effect on the yield curve. A more formal test, represented in Diebold et al. (2006), also supports this view

5.2.3 Structural model by Lemke (2008)

The essential goal of Lemke's (2008) structural model was to examine how a shock to macro variables affects TS. Although interesting, the data used is quarterly and the sample period is short, which means that the actual number of observation is rather low, which probably does not lead to trustworthy empirical results.

The macroeconomic framework is based on a structural model that is closely related to the work of Laubach & Wiliams (2003) and Mesonnier & Renne (2006). In the Lemke (2008) model, a backward-looking Phillips curve explains the dynamics of inflation, while a backward-looking IS equation demonstrates the dynamics of the output gap and is also one of the explanatory variables of the Phillips curve. Furthermore, the joint dynamics of potential output growth and natural rate of interest rate is part of the model describing the dynamics of the macroeconomy (Lemke, 2008). All of the equations and related dynamics in Lemke (2008) are presented here:

$$(13)\ \ \pi_{t+1} = c_\pi + \alpha_1\pi_t + \alpha_2\pi_{t-1} + \alpha_3\pi_{t-2} + \beta z_t + \epsilon^{\pi}_{t+1}, where \sum_{j=1}^{3} a_j < 1,$$

$$(14)\ \ z_{t+1} = \psi_z z_t + (1+L)\gamma(i_t - E_t(\pi_{t+1}) - r^*_t) + \epsilon^{z}_{t+1},$$

$$(15)\ \ r^*_t = c_r + \theta_r a_t,$$

$$(16)\ \Delta y_t^* = c_y + \theta_y a_t + \epsilon_t^y,$$

$$(17)\ \ a_{t+1} = \psi_a a_t + \epsilon_{t+1}^a,$$

$$(18)\ y_t = y_t^* + z_t,$$

$$(19)\ i_t = \phi_i i_{t-1} + (1 - \phi_i)\big(c_i + \phi_\pi \pi_t + \phi_g \Delta y_t\big) + v_t,$$

$$(20)\ v_t = \psi_v v_{t-1} + \epsilon_t^v.$$

Equation (13) describes the Phillips curve in logarithmic form. In this case, this depends on: the value of a constant[46] c_π, lags of inflation itself, the previous period's output gap, and the supply side cost-push shock ϵ_t^π. The parameter β defines the impact that the last period's output gap brings to the current inflation (Lemke, 2008).

Equation (14) presents the output gap dynamics in logarithmic form. Here, L is the lag operator[47] and $i_t - E_t(\pi_{t+1}) - r_t^*$ is the nominal one-quarter interest rate minus expected inflation (derived from the model estimation) minus the equilibrium interest rate (Lemke, 2008). This describes whether the real interest rate (nominal interest rate minus inflation expectation) is above or below the equilibrium real interest rate, r_t^*. This interest rate is also known as the neutral real interest rate (NRI) and has gained a lot of attention as part of the New Keynesian framework (Amato, 2005). As the parameter γ (which describes the weight given to the real interest rate gap) is negative, a high real interest rate implies a decrease in the output gap and inflation. Idiosyncratic supply side shock ϵ_t^z is also part of output gap determination (Lemke, 2008).

Equations (15) and (16) share a common parameter, a_t, the trend growth rate of output whose dynamics are given in (17) (Lemke, 2008). The equilibrium interest rate constitutes a constant part c_r and an estimated parameter, θ_r, that describes the estimated weight given to the trend growth rate, a_t, in the determination of equilibrium interest rate r_t^* (Lemke, 2008). Potential output growth, Δy_t^*, has been defined by a constant c_y trend growth rate and transitory (temporary) shock, ϵ_t^y. The actual log output, y_t, is given in (18); that is, the sum of log

[46] This constant ensures a non-zero unconditional mean for inflation as, by definition, the expected value of the output gap z_t is zero.

[47] The only purpose of lag operators is to shorten the notation. Without it, the second part of the right-hand side in equation (8) would be $\gamma(i_t - E_t(\pi_{t+1}) - r_t^*) + \gamma(i_{t-1} - E_{t-1}(\pi_t) - rt-1*$. A general example from the use of the lag operator is $Lyt=yt-1$ from Verbeek (2008).

potential output y_t^* and output gap z_t. This actually means that the output gap is defined by deducting the log potential output from the actual log output.

As in many other MTS models, the nominal interest rate (policy rate) in this model is defined endogenously (19) and is determined by constant, c_i, instantaneous inflation, π_t, output growth, $g_t \equiv \Delta y_t (= y_t - y_{t-1})$, and monetary policy shock, v_t, which is consistent with the estimated autocorrelation of 0.97 (Lemke, 2008). The market price of risk is the same as in the HTV model $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1 \boldsymbol{X}_t$ except that vector $\boldsymbol{\lambda}_1$ is set to zero; otherwise it would deliver unreliable results due to the small sample space (Lemke, 2008). This means that the market price of risk is a constant, unlike in HTV where it was time varying. Lemke pointed out that in several other studies with longer data periods, the market prices of risk $\boldsymbol{\lambda}_t$ are, and should be, time varying. However, given that Lemke's bond yield data was from 1998 to 2006, it does not comprise enough observations to make the empirical results valid. The macroeconomic data, on the other hand, starts from 1981 to 1999 and uses hypothetical Euro area data during this period. This type of restructuring has also been used by Gerdesmeier & Roffia (2004) and Gerlach & Kristen (2003). However, because the aim is to get some insight into the TS formation, this longer sample period for macro variables does not make a difference to the validity of the empirical results.

In the Lemke (2008) macro model described above, the free parameters are the constants c_y, c_r, c_π, c_i, lagged inflation coefficients $\alpha_1, \alpha_2, \alpha_3$, output gap coefficient in the Phillips curve β, autoregressive parameters $\psi_z, \psi_a, \phi_i, \psi_v$, the effect of output growth on equilibrium interest rate, and potential output growth θ_r, θ_y. All five shocks have been determined to be uncorrelated and normally distributed and the variance of trend growth rate σ_a^2 has been normalised to unity, while the remainder of the shocks $\sigma_\pi, \sigma_z, \sigma_y, \sigma_v$ are free parameters. Also, the market prices of risk are free parameters, namely $\lambda_{0\pi}, \lambda_{0a}, \lambda_{0z}, \lambda_{0y}\lambda_{0v}$, and the rest of the free parameters defined earlier are γ, ϕ_π, ϕ_g.

The unobserved variables, that is, the state variables, constitute the state vector $\boldsymbol{X}_t = (\pi_t, \pi_{t-1}, \pi_{t-2}, \pi_{t-3}, g_t, i_t, i_{t-1}, a_t, a_{t-1}, z_t, z_{t-1}, v_t)^T$ (Lemke, 2008). The state space model has been derived in Lemke (2008) from this and other aforementioned equations. The actual bond pricing procedure is based on the arbitrage-free framework and the stochastic discount factor, as in the HTV model. It is quite difficult to compare the Lemke (2008) model and the

HTV model from an empirical point of view because the yield data is only from nine years (quarterly) in Lemke, while HTV uses data from 24 years (monthly). Despite this, the models are quite alike, although Lemke is unable to constitute a full model of market prices of risk due to the short time period. Furthermore, Lemke uses a constant inflation target, which is actually well suited for the 1998–2006 time period, as the ECB uses quite fixed inflation target.[48] Lemke also defined the natural real interest rate to be time-varying, which seems plausible as the nominal interest rate is much more volatile than the inflation rate. In the Lemke model, the natural real interest rate follows an autoregressive process instead of the more commonly used random walk. This hypothesis is well justified in Mésonnier & Renne (2007), which provides the reasoning for using the autoregressive process. All in all, HTV relies more on model-determined expectations while the Lemke model uses more backward-looking elements in the modelling.

The estimation in Lemke (2008) is done in three phases. The first is the calibration in which the annualised potential output growth, c_y, and the long-run natural interest rate, c_r, have been estimated without the macro model. In addition, some other variables have been estimated in the calibration phase based on the model characteristics by using unconditional expectations. In the second phase, the macro variables were estimated using the Kalman filter in order to maximise the likelihood function. The final phase focused on the estimation of the market price of risk parameters[49] via the maximum likelihood procedure (Lemke, 2008). However, like many others, Lemke noted that if all the market prices of the risk parameter would be estimated, the results would be statistically insignificant. Hence, only three market prices of risk parameters have been estimated, corresponding to inflation ϵ^{π}, trend growth rate ϵ^{a}, and monetary policy ϵ^{v} shocks (Lemke, 2008). These three shocks were chosen because they contributed the greatest variation in yields according to variance decomposition shown in Lemke (2008). The monetary policy shock is accountable for most of the variation for shorter forecast horizons and for shorter yields and decreases as forecast horizon or yield increases. The trend growth rate reacts in the opposite way, increasing as the forecast horizon or the applied yield increases.

[48] The official target is below two percent p.a. Before ECB, the German Bundesbank was also known to focus on keeping inflation under control.
[49] These fixed parameters have been estimated for each source of uncertainty in the economy. In the Lemke model, this means five market price of risk estimates, for: inflation π, trend growth rate a, IS equation (14) z, and potential output growth Δy_t^*.

Although the Lemke (2008) study was theoretically interesting, its low number of observations meant that it lacked empirical validity.

Attention now shifts to a totally different framework, the DSGE model, which aims to give empirical results that are at least as good as those of the HTV model. The DSGE model has succeeded in the empirical part and does model the macroeconomy with more theoretical rigour than the other structural models above. Because of these attributes, many scholars find the DSGE model to be the next great thing when it comes to the modelling of the macroeconomy as a whole.

5.3 The DSGE macro model as part of TS modelling

The DSGE model (also a structural model) of the macroeconomy is used by the European Central Bank (ECB) to evaluate the Eurozone as one entity. Their model was named after its developers, Smets-Wouters (2003).[50] This model has attracted criticism from, for example Gregory Mankiw,[51] considered one of the developers of the so-called New-Keynesian DSGE model. One of the "loudest" critiques of the DSGE model was provided by Willem Buiter, who writes provocatively in a blog[52] for the Financial Times and also finds the New-Keynesian and New-Classical theories quite misleading for describing the macroeconomy. However, most economists have found the DSGE model to be quite an adaptive and able form of modelling (otherwise, the ECB would not have adopted this model).

The DSGE model, as part of TS modelling, was developed by De Graeve et al. (2009), probably the first to incorporate the DSGE model and TS of interest rates. One of the reasons why De Graeve et al. (2009) decided to conduct the study could have been that the predictive power had been increased in previous MTS models compared to the VAR models using additional degrees of freedom, which was a result of using more flexible modelling, such as the latent variables in Ang & Piazzesi (2003) or time-varying variances of structural shocks in

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[50] The ECB link to this document is http://www.ecb.int/home/html/researcher_swm.en.html, from which the original paper may also be downloaded.
[51] Mankiw argued that there are several shortcomings in the New-Keynesian DSGE but still found it important for the development of new models (Mankiw, 2006).
[52] The address to the Maverecon blog is http://blogs.ft.com/maverecon/2009/03/the-unfortunate-uselessness-of-most-state-of-the-art-academic-monetary-economics/. The insightful text was read on the 14th of June.

Doh (2007). This flexibility has aroused concerns that the models presented above have inadequate formation of the underlying macroeconomy.

The strength of the DSGE TS model by De Graeve et al. (2009) is built on a more detailed representation of macroeconomy as it builds up the macro model via micro foundations. Indeed, the DSGE provides quite a comprehensive description of the macroeconomy, where the economy in De Graeve et al. (2009) consists of households, final and intermediate goods firms and monetary authority. The general setup is that consumers provide differentiated labour to a monopolistically competitive labour market. Consumers are the owners of capital stock; they decide on investments and rent capital services to companies. The utility of a consumer is completely explained by their contribution to consumption and labour force (De Graeve et al., 2009).

The transition equation has been formed by 13 equations that explain how the DSGE model evolves in time. Four equations are needed for the aggregate demand and the other nine equations are for the aggregate supply side. The AD part consists of aggregate resource constraint, consumption, investments and current value of capital. The AS is formed by aggregate production, current capital services used in production, degree of capital utilisation, accumulation of installed capital, New-Keynesian Phillips curve, firm's marginal cost, rental rate of capital, real wage and monetary policy.

The DSGE model follows microeconomics, as the fundamental assumptions on agents' preferences and technologies are solved by using intertemporal optimisation. The next section describes the dynamics of AD in De Graeve et al. (2009).

5.3.1 Aggregate demand on DSGE

The aggregate resource constraint explains the output $\hat{y}_t$ via consumption $\hat{c}_t$, investment $\hat{\imath}_t$, exogenous spending $\hat{\varepsilon}_t^g$, and capital utilisation rate $\hat{z}_t$. The equation for the aggregate resource constraint is:

$$(21)\ \hat{y}_t = \hat{\varepsilon}_t^g + \frac{c_*}{y_*}\hat{c}_t + \frac{i_*}{y_*}\hat{\imath}_t + \frac{r_*^k k_*}{y_*}\hat{z}_t.$$

The starred variables in this equation denote steady state values and $r_*^k k_*$, the product of capital rental rate and capital stock, respectively. The exogenous spending follows an AR(1) process[53] with an IID-Normal error that is explained, among other things, by a (ϵ_t^a) productivity shock (see footnote 54) (De Graeve et al., 2009). The productivity shock captures the relation between net exports, including the exogenous spending, which might have an effect on domestic productivity developments (De Graeve et al., 2009).

The consumption dynamics depends on past consumption, on the expected future consumption and the expected change in hours worked $\left(E_t\left[\hat{L}_{t-1}\right] - \hat{L}_t\right)$, which has an effect on current consumption. The ex-ante real interest rate was defined as $\left(\hat{R}_t - E_t[\hat{\pi}_{t+1}]\right)$ and the disturbance term by $\hat{\varepsilon}_t^b$. The consumption dynamics was given by:

$$\hat{c}_t = \frac{1}{\left(1 + (\lambda/\gamma)\right)} E_t[\hat{c}_{t+1}] + \frac{(\lambda/\gamma)}{\left(1 + (\lambda/\gamma)\right)} \hat{c}_{t-1} - \frac{(1 - \lambda/\gamma)}{\sigma_c\left(1 + (\lambda/\gamma)\right)} \left(\hat{\varepsilon}_t^b + \hat{R}_t - E_t[\hat{\pi}_{t+1}]\right) - \frac{(\sigma_c - 1)(w_*^h L_*/c_*)}{\sigma_c\left(1 + (\lambda/\gamma)\right)} \left(E_t\left[\hat{L}_{t+1} - \hat{L}_t\right]\right),$$

This follows the consumption Euler equation,[54] in which the structural parameters γ, λ, and σ_c measure the trend growth rate, the degree of habit persistence and risk aversion, respectively (De Graeve et al., 2009). The disturbance term $\hat{\varepsilon}_t^b$ works as a wedge between the policy interest rate controlled by the central bank and the return on assets held by households, as shown by Chari et al. (2007). A positive shock on consumption increases the required return on assets and reduces current consumption as a larger part of capital held by households goes to investments rather than consumption. The disturbance term $\hat{\varepsilon}_t^b$ also increases the cost of capital, which means that the value of capital and investments decreases when the discount rate increases (De Graeve et al., 2009).

The dynamics of investment follows the investment Euler equation:

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[53] The exact process is given by: $\hat{\varepsilon}_t^g = \rho_g \hat{\varepsilon}_t^g + \rho_{ga} \epsilon_t^a + \epsilon_t^g$ (De Graeve et al., 2009).

[54] The basic Euler equation for consumption states that the current utility from consumption now should equal the discounted consumption later. If there is a difference between these two, consumption now and later one should adjust consumption accordingly. In the consumption Euler equation given here, the model is somewhat more sophisticated, which allows the previous period’s consumption, inflation expectation and hours worked to have an effect on the equation.

$$\hat{\imath}_t = \frac{1}{(1+\bar{\beta}\gamma)}\left[\hat{\imath}_{t-1} + (\bar{\beta}\gamma)\hat{\imath}_{t+1} + \frac{1}{\gamma^2 S''}\hat{Q}_t^k\right] + \hat{\varepsilon}_t^I.$$

Here the investment is defined as being a function of $\bar{\beta} = (\beta/\gamma^{\sigma_c})$, where β is the discount factor used by households and γ^{σ_c} is the trend growth rate to the power of risk aversion. S'' describes how the steady-state elasticity of the capital adjustment cost changes when $\bar{\beta}$ changes. $\hat{Q}_t^k$ is the real value of capital. According to the investment equation, the real value of capital has less influence on the investment as the trend growth rate γ and S'' become higher. The last term, $\hat{\varepsilon}_t^I$, describes a disturbance in the investment composition. This disturbance term follows the AR(1) process with IID$\sim N$ error term ϵ_t^I according to $\hat{\varepsilon}_t^I = \rho_I \hat{\varepsilon}_{t-1}^I + \epsilon_t^I$ (De Graeve et al., 2009).

The final equation on the AD side is the real value of existing capital stock and is given as:

$$\hat{Q}_t^k = -\hat{\varepsilon}_t^b - \left(\hat{R}_t - E_t[\hat{\pi}_{t+1}]\right) + \frac{r_*^k}{r_*^k + (1-\delta)} E_t[\hat{r}_{t+1}^k] + \frac{(1-\delta)}{r_*^k + (1-\delta)} E_t\left[\hat{Q}_{t+1}^k\right].$$

Here the current value of real capital stock is positively dependent on the expected real rental rate on capital $E_t[\hat{r}_{t+1}^k]$ and its own expected future value $E_t\left[\hat{Q}_{t+1}^k\right]$. δ denotes the depreciation rate and $\hat{\varepsilon}_t^b$ is the disturbance term of aggregate demand (De Graeve et al., 2009). The ex-ante real interest rate, $\hat{R}_t - E_t[\hat{\pi}_{t+1}]$ and $\hat{\varepsilon}_t^b$ have a negative impact on capital stock (De Graeve et al., 2009), which means that the larger the difference between the nominal interest rate $\hat{R}_t$ and expected inflation in the next period $E_t[\hat{\pi}_{t+1}]$, the lower the capital stock (unless the real interest rate is negative). The higher the ex-ante real interest rate, the fewer investments are profitable; in other words, the expected real rental rate is required to be higher in order to compensate for the higher ex-ante real interest rate.

5.3.2 Aggregate supply on DSGE

Having provided a summarised representation of the issues that have an effect on AD formation, the thesis now shifts its attention to some dynamics of AS determination in the DSGE framework. It introduces the supply side that provides the goods and services to a given economy. This starts with an introduction of the aggregate production function that describes how the production works in optimal situation; that is, providing the maximum output with given inputs as follows:

$$\hat{y}_t = \Phi\left(\alpha \hat{k}_t^s + (1-\alpha)\hat{L}_t + \hat{\varepsilon}_t^a\right).$$

In this equation, the production of output $\hat{y}_t$ requires capital, $\hat{k}_t^s$, and labour services, $\hat{L}_t$, which are measured as hours worked (De Graeve et al., 2009). Capital and labour services are the inputs while total factor productivity is denoted by $\hat{\varepsilon}_t^a$ (De Graeve et al., 2009), which accounts the effects on output not picked by the aforementioned inputs, such as exceptionally dry weather in a country that is substantially depended on agricultural output. The total factor productivity follows the AR(1) process according to $\hat{\varepsilon}_t^a = \rho_a \hat{\varepsilon}_{t-1}^a + \epsilon_t^a$. As before, the error term ϵ_t^a is IID-Normal (De Graeve et al., 2009).

Current (at time *t*) capital services, $\hat{k}_t^s$, used in production becomes effective after a one-quarter lag and is a function of previous periods installed capital, $\hat{k}_{t-1}$, and the degree of capital utilisation, $\hat{z}_t$. The capital services and degree of capital utilisation have been given by the two following equations, respectively:

$$\hat{k}_t^s = \hat{z}_t + \hat{k}_{t-1}; \quad \hat{z}_t = \frac{1-\psi}{\psi} \hat{r}_t^k.$$

Here, the degree of capital utilisation is explained by ψ, the elasticity of the capital utilisation adjustment cost function, which is normalised as between zero and one[55] (De Graeve et al., 2009). In a situation where ψ is close to one, the adjustment of capital is very costly. On the other hand, when ψ is close to zero, the adjustment cost of capital is very low.

De Graeve et al. (2009) represented several other equations[56] but, in this context, the full representation of these equations would lead to incorrect emphasis. Instead, the representation of models, fitted to the in-sample data following the out-of-sample performance, is more interesting.

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[55] It cannot be exactly zero as the degree of capital utilisation could not be defined in that case.

[56] These equations include: New-Keynesian Phillips curve, marginal cost, real wage and monetary policy reaction function (De Graeve et al., 2009).

5.3.3 DSGE model prediction performance

In-sample, the empirical ability of the De Graeve et al. (2009) DSGE model performs quite well. The size of the standard deviation of the measurement errors for yields varies between the 17 and 32 basis point in annual terms, 17 for the three-year yield and 28 for the one-year yield. These figures are close to that of HTV, for which the standard deviation of the measurement errors varies from 23 to 28 basis points for the in-sample estimation. These estimates are suitable for comparison as the estimation period is rather long for both models: 1966:1 to 2007:1 for the De Graeve et al. (2009) model, based on quarterly observations, and 1975:1 to 1998:12 for the HTV model, based on monthly observations. The difference between these two data sets is that De Graeve et al. (2009) is based on US data, while the HTV model is based on German data. Although this presents some obvious problems, these results should still be seen as strengthening the viability of the DGSE model.

In the paper by De Graeve et al. (2009), the out-of-sample forecast for the yields of varying maturities and forecast periods do not outperform the Random Walk hypothesis (except in a few cases). This result can be seen as evidence that the more flexible HTV model might lead to better forecast results. However, the HTV model has been a result of extensive research in the area of flexible MTS models, while De Graeve et al. (2009) is one of the first models – if not the first – to connect the DSGE macro modelling with the term structure of interest rate. It is quite probable, therefore, that the DSGE model framework could be used more extensively in the near future than it is today. Before that occurs, however, the next section provides some remarks about the suitability of the DSGE model to TS modelling.

5.3.4 Conclusion of the DSGE model applicability to term structure modelling

The DSGE macro models are, in general, based on microeconomic foundations and, as such, have strong theoretical foundations. As the De Graeve et al. (2009) model provides more restricted and uniform structuring of the macroeconomics; it will most probably be studied in the term structure context more and more extensively in the future.

It is almost certain that in the future there will be empirically more accurate models and that the “competition” between theoretically and empirically coherent models will continue. For the time being, it seems that in the quite short history of implementing macroeconomy to the TS there has been a tremendous evolution of models and it may well be that the DSGE models will take an increasing proportion of these new models. This continuum, from purely mathematical models towards economically coherent models, has clearly proved within the 21st century that the TS is largely explained by macro variables.

6 CRITICISM OF THE MTS MODELS

The MTS models are quite new and still very much evolving towards more rigorous versions. One can expect, therefore, that the problems these models have will be quite different in the near future. This section reveals the problems of MTS models, most of which are related to macroeconomic modelling.

The basic idea supporting the use of the macroeconomy in the context of TS modelling is quite simple: the central banks' policy rules. These rules are based on several macroeconomic indicators and many of the central banks have published this in their policies. It is quite obvious that the policy interest rate has an effect on market interest rates and on bond pricing, so the fact that the macroeconomy has a clear causal relationship with the TS cannot be ignored.

Although the MTS models all rely on the macroeconomy, there is a significant difference between a VAR model and a DSGE model. Theoretically, DSGE models should be seen as more valid means of studying the TS, as the macroeconomy developed in the DSGE model is more rigorously modelled (De Graeve et al., 2009). De Graeve et al. (2009) was the first to introduce the DSGE framework to TS modelling and brought up the fact that the recent and more flexible VAR models have brought the implied yields and observed yields closer together. However, this empirical success does not necessarily mean that these models would not be misspecified. De Graeve et al. (2009) argued that the more accurate empirical results of these flexible VAR models might be the result of higher degrees of freedom, which are inevitable when the various flexible features are brought in. However, this argument should not be valid when comparing the out-of-sample results.

One of the most widely used and accepted constraints is the no-arbitrage assumption. It is true that some bonds might not be as liquid as required by the no-arbitrage theory and, also, the misspecification of the underlying model would degrade the empirical performance of a model that uses the no-arbitrage condition (Diebold et al., 2005). However, Ang & Piazzesi (2003) presented convincing empirical evidence in favour of imposing no-arbitrage restrictions on TS models. Accordingly, this assumption has its pros and cons but, for the time being, the no-arbitrage constraint has been widely used.

More controversial but still widely used is the assumption that yields are affine in the state variables. The reason why this assumption is widely in use is that it makes many otherwise cumbersome issues easier to solve. However, there are also some problems with this property. If the number of bond yields in a data set at one point in time exceeds the number of state variables, as it usually does, extra error term(s) have to be added to the model (Campbell et al., 1997). These error terms, in turn, decrease the explanatory power of the model. Another issue is that the affine-yield models restrict the way in which interest rate volatility can change with the level of interest rates. For example, a model in which the volatility of the interest rate is proportional to the cube root of the interest rate is not affine and is therefore unacceptable in the affine framework (Campbell et al., 1997).

The most significant argument that favours the MTS model is the empirical results from the out-of-the-sample forecast, which support the fact that incorporating macroeconomic theory to statistics and finance theory can and does deliver more profound results than when these disciplines are used separately.

As mentioned before, the most efficient way to improve the MTS models is to model the macroeconomy more rigorously, as was done in the DSGE model. Following the recent downturn in the global economy, many arguments against the New Keynesian and New Classical theories of macroeconomics have been presented. These theories, especially the New Keynesian one, have been widely used by central banks and other institutions. The New Keynesian theory is also one of the main building blocks behind macroeconomic modelling in the MTS models presented above; the only exception is the DSGE model, which can actually be seen as a mixture of New Keynesian and New Classical theories.

In his Maverecon blog, published by Financial Times, Willem Buiter alleged that there are several shortcomings in the New Keynesian and New Classical theories, as well as in the DSGE model. The main message is that all three of these macroeconomic theories are based on overly unrealistic assumptions. However, these arguments are targeted at the more sophisticated macromodels in the New Keynesian and New Classical framework, which are not used in the TS models described earlier, except in the DSGE model (Spahn, 2009). In fact, there is an established consensus that the TS models that use New Keynesian framework are actually quite accurate abstractions of real life (Spahn, 2009). However, these models are not good enough to yield accurate forecasts.

The major shortcoming of the DSGE model used in De Graeve et al. (2009) is that agents are assumed to understand the underlying model. Furthermore, it is assumed that all agents are alike, that is, that they all have the same information set, which includes the information of the model. Thus, the DSGE model is useful only when making forecasts for very long time periods (De Grauwe, 2008).

Unfortunately, agents are not the same in real life and the information set used definitely varies between agents. Instead, agents use “simple rules” to help their decision-making process when facing very complex problems. This behaviour is rational, as everyone’s ability to understand the behaviour of human interactions and the outcome of it is limited (De Grauwe, 2008).

De Grauwe (2008) presented a model in which agents use simple rules to forecast future phenomena. This type of DSGE model is particularly interesting from the point of view of TS modelling. However, the model in question has not yet been empirically tested or augmented with the TS.

It seems once again that a severe economic shock was needed in order to give momentum to new ideas. The development of new MTS models has been extremely rapid. Even the latest macroeconomic models, such as the DSGE framework, have already been used as part of the TS modelling. The shortcomings of the New Keynesian and New Classical models have been exposed and many scholars seem to be thinking of new ways to model the macroeconomy. A great example of this exploration is De Grauwe (2008).

It is surprising that the amount of money in circulation has not been incorporated into the Taylor or Wicksell rules, outlined above. It is clear that, at least in Greece, the current buying spree of toxic bonds by the ECB has an influence on interest rate formation. It is well known that the amount of M3 money, for example, has an effect on inflation, which in turn is the most significant determinant for ECB when setting the policy interest rate for the Euro area.

This could well be one of the missing links in the pool of explanatory variables for the determination of TS.

7 CONCLUSION AND THOUGHTS FOR FUTURE STUDY

The goal of this thesis was to provide extensive coverage of the current TS models, with emphasis on the macro variables and models. This journey started by introducing some of the historically important research by Vasicek (1977) and another by Cox et al. (1985), which was the first to model the matter of uncertainty of TS well without the complication of the macroeconomy. The next step was to introduce Wicksell's policy interest rate rule, and Taylor's better known and more widely used model was also introduced. The basic presentation for most current MTS models is the state space model, which followed the policy interest rate rules in the thesis. An example of a factor model was presented, before moving into the more advanced models that incorporate TS macro models. These models were categorised into three – the VAR model, the Structural model and the DSGE model – according to how the macroeconomy was modelled. The last part was devoted to criticism.

This method of organising the literature review was logical and supported the aim of introducing the latest developments in the TS modelling, in which the macro variables have become essential explanatory variables.

The two most important findings were made by De Graeve et al. (2009), in their DSGE model, and its modification by De Grauwe (2008). These two models based their macro model on micro-foundations that are, theoretically, the most valid. De Grauwe (2008) introduced the idea of agents using simple rules when forming future expectations; this idea is new in the DSGE context and deserves extra attention as it has not yet been empirically tested. Also, it has never been linked to the TS modelling, which is unfortunate.

Both of the models mentioned in the previous paragraph are based on a very new DSGE macromodel, so it is no surprise that many of the latest discoveries in the MTS models have been motivated by a new invention in macro modelling. However, the dominant model family – the structural models – have relied on flexibility that has enabled more accurate empirical results than before. The weakness of these models is the loose macroeconomic framework, as well as the flexibility of other factors, such as additional shocks, which increase the degree of freedom that will reduce the validity of these structural models. It might be, therefore, that the "competition" between the structural and DSGE models becomes even greater in the near future. The DSGE models have proven to be more theoretically valid and at least as empirically valid as these structural models that incorporate flexibility in order to gain better empirical fit to the data.

These new developments in TS modelling have given the Central Banks many new ways to approach the problem of dampening the economic cycles by governing the policy interest rate and the amount of money in circulation. With the current uncertainty in the global economy, it is very important for the Central Banks' to use the latest research results, together with older ones, in order to make the best possible future forecasts.

For the time being, it seems that the most interesting research in the near future involves the DSGE model and various explanatory variables, of which the inflation and latest observed yields seem to be the most influential in the long run. In the short run, it seems that the random walk hypothesis is indeed very difficult, if not impossible, to beat. It is unsurprising that the macro variables do not perform very well in the short run, as the data from the macroeconomy is more or less from the past. This is quite an excuse, however, as the daily fluctuation is most probably on real events rather than a totally random process. Still, it might be the case that the random processes, such as the random walk, are the best proxy in the short run, while MTS models should be favoured for the longer run.

8 BIBLIOGRAPHY

Amato, J. (2005). *The role of the natural rate of interest in monetary policy.* Bank of Internation Settlements.

Ang, A., & Bekaert, G. (2004). *The Term Structure of Real Rates and Expected Inflation.* C.E.P.R.

Ang, A., & Piazzesi, M. (2002). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of monetary economics , 50* (4), 745-787.

Ang, A., Piazzesi, M., & Wei, M. (2006). What does the yield curve tell us about GDP growth? *Journal of econometrics , 131* (1-2), 359-403.

Brousseau, V. (2002). *The Functional Form of Yield Curves.* European Central Bank.

Campbell, J. Y. (1995). Some lessons from the yield curve. *Journal of Economic Perspectives* , 129-152.

Campbell, J., Lo, A., & MacKinlay, A. (1997). *The econometrics of financial markets.* New Jersey: Princeton University Press.

Chan, K. C., Karolyi, G. A., Longstaff, F. A., & Sanders, A. B. (1992). An Empirical Comparison of Alternative Models of the Short-Term Interest Rate. *Journal of Finance , 47* (3), 1209-1227.

Chari, V., Kehoe, P., & McGrattan, E. (2007). Business cycle accounting. *Econometrica , 75* (3), 781-836.
Chatfield, C. (2004). *The analysis of time series: an introduction.* Florida: CRC Press.
Christensen, J., Diebold, F., & Rudebusch, G. (2007). *The affine arbitrage-free class of Nelson-Siegel term structure models.* Federal reserve bank of San Francisco.
Christiano, L., Eichenbaum, M., & Evans, C. (2001). *Nominal rigidities and the dynamic effects of a shock to monetary policy.* Cleveland: Federal Reserve Bank of Cleveland.
Christiano, L., Eichenbaum, M., & Evans, C. (1996). The Effects of Monetary Policy Shocks: Evidence from the Flow of Funds. *Review of Economics & Statistics , 78* (1), 16-34.
Clarida, R., Galí, J., & Gertler, M. (2000). Monetary policy rules and macroeconomic stability: evidence and some theory. *Quarterly Journal of Economics , 115* (1), 147-180.
Clarida, R., Galí, J., & Gertler, M. (1998). Monetary Policy Rules in Practice: Some International Evidence. *European Economic Review* , 1033-1067.
Cochrane, J. H. (2001). *Asset pricing.* New Jersey: Princeton University Press.
Cox, J. C., Ingersoll, J., & Ross, S. (1985). A theory of the term structure of interest rates. *Econometrica , 53* (2), 385-407.
Cox, J., Ingersoll, J., & Ross, S. (1981). The relationship between forward prices and futures prices. *Journal of Financial Economics , 9* (4), 321-346.
Dai, Q., & Singleton, K. (2000). Specification Analysis of Affine Term Structure Models. *The Journal of Finance , 55* (5), 1943-1978.
De Graeve, F., Emiris, M., & Wouters, R. (2009). A structural decomposition of the US yield curve. *Journal of monetary economics , 56* (4), 545-559.
De Grauwe, P. (2008, May). Macroeconomic modelling when Agents are Imperfectly Informed. *Macroeconomic modelling when Agents are Imperfectly Informed* . Leuven, Belgium: European Central Bank.
Dewachter, H., & Lyrio, M. (2006). Macro Factors and the Term Structure. *Journal of money, credit, and banking , 38* (1), 119-140.
Diebold, F. X., Piazzesi, M., & Rudebush, G. D. (2005). Modeling Bond Yields in Finance and Macroeconomics. *American economic review , 95* (2), 415-420.
Diebold, F., Rudebusch, G., & Aruoba, S. (2006). The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of econometrics , 131* (1-2), 309-338.
Doh, T. (2007). *What moves the yield curve? Lessons from an estimated nonlinear macro model.* University of Pennsylvania.
Dougherty, C. (2007). *Introduction to econometrics.* New York: Oxford univeristy press .
Duffee, G. R. (2002). Term Premia and Interest Rate Forecasts in Affine Models. *The Journal of Finance , 57* (1), 405-443.
Duffie, D., & Kan, R. (1996). A Yield-Factor Model Of Interest Rates. *Mathematical Finance , 6* (4), 379-406.
Durbin, J., & Koopman, S. J. (2001). *Time series analysis by state space methods.* Oxford: Oxford University Press.
Estrella, A., & Hardouvelis, G. (1991). The Term Structure as a Predictor of Real Economic Activity. *Journal of Finance , 46* (2), 555-576.
Estrella, A., & Mishkin, F. (1997). The predictive power of the term structure of interest rates in Europe and the United States: Implications for the European Central Bank. *European Economic Review , 41* (7), 1375-1401.
Evans, C., & Marshall, D. (1998). Monetary policy and the term structure of nominal interest rates: Evidence and theory. *Carnegie-Rochester Conference Series on Public Policy , 49* (1), 53-111.
Evans, M. (2003). Real risk, inflation risk and the term structure. *The Economic Journal , 113* (487), 345-389.

Gerdesmeier, D., & Roffia, B. (2004). Empirical estimates of reaction functions for the euro area. *Swiss journal of economics and statistics , 140* (I), 37-66.
Gerlach-Kristen, P. (2003). *Interest rate reaction functions and the Taylor rule in the euro area.* European Central Bank.
Gürkaynak, R. S., Sack, B., & Swanson, E. (2003). The excess sensitivity of long-term interest rates: evidence and implications for macroeconomic models. *Division of Monetary Affairs Board of Governors of the Federal Reserve System* (pp. 1-44). Washington DC: The Federal Reserve .
Hall, R. (1978). Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence. *Journal of Political Economy , 86* (6), 971-987.
Hein, E., & Truger, A. (2007). *Money, distribution and economic policy: alternatives to orthodox macroeconomics* . Edward Elgar Publishing Limited.
Hördahl, P., Tristani, O., & Vestin, D. (2006). A joint econometric model of macroeconomic and term-structure dynamics. *Journal of econometrics* , 405-444.
John, H. (1939). *Value and Capital: An Inquiry into Some Fundamental Principles of Economic Theory.* Oxford: Clarendon Press.
Kozicki, S., & Tinsley, P. (2002). Dynamic Specifications in Optimizing Trend-Deviation Macro Models. *Journal of economic dynamics and control , 26* (9-10), 1585-1611.
Kozicki, S., & Tinsley, P. (2001). Shifting Endpoints in the Term Structure of interest rates. *Jurnal of monetary economics , 47* (3), 613-652.
Laubach, T., & Williams, J. (2003). Measuring the natural rate of interest. *Review of Economics and Statistics , 85* (4), 1063-1070.
Lemke, W. (2008). An affine macro-finance term structure model for the euro area. *North American journal of economics and finance* , 41-69.
Litterman, R., & Scheinkman, J. (1991). Common factors affecting bond returns. *Journal of fixed income* , 51-61.
Lutz, F. (1940). The Structure of interest rates . *Quarterly Journal of Economics* , 36-63.
Lütkepohl, H. (1991). *Introduction to multiple time series analysis.* Berlin: Springer-Verlag.
Malkiel, B. G. (1966). *Term Structure of Interest Rates: Expectations and Behavior Patterns.* New Jersey: Princeton University Press .
Mankiw, N. G. (2006). The Macroeconomist as Scientist and Engineer. *Journal of Economic Perspectives , 20* (4), 29-46.
Mésonnier, J.-S., & Renne, J.-P. (2007). A time-varying ‘‘natural’’ rate of interest for the euro area. *European economic review* , 1768-1784.
Nelson, C. R., & Siegel, A. F. (1987). Parsimonious Modeling of Yield Curves. *Journal of business , 60* (4), 473-489.
Piazzesi, M. (2010). Affine term structure models. In Y. Ait-Sahalia, & L. Hansen (Eds.), *Handbook of financial econometrics* (pp. 691-766). North Holland, Elsevier.
Rudebush, G. D. (2002). Assessing Nominal Income Rules for Monetary Policy with Model and Data Uncertainty. *The economic journal* , 402-432.
Sargent, T. (1979). A note on maximum likelihood estimation of the rational expectations model of the term structure. *Journal of Monetary Economics* , 245-274.
Shreve, S. E. (2004). *Stochastic calculus for finance II, continuous-time models.* New York: Springer.
Spahn, H.-P. (2009, August). The New Keynesian Microfoundation of Macroeconomics. *The New Keynesian Microfoundation of Macroeconomics* . Stuttgart, Germany: University of Hohenheim.
Sundaresan, S. M. (2002). *Fixed Income Markets and Their Derivatives.* Ohio: South-Western for Thomson Learning.

SØrensen, P. B., & Whitta-Jacobsen, H. J. (2005). *Introducing advanced macroeconomics: growth & business cycles.* Berkshire: McGraw-Hill Education.
Tashman, L. J. (2000). Out-of-sample tests of forecasting accuracy: an analysis and review. *International journal of forecasting , 16* (4), 437-450.
Taylor, J. (1993). Discretion versus policy rules in practice. *Carnegie-Rochester Conference Series on Public Policy* , 195-214.
Vasicek, O. (1977). An equilibrium characterization of the term structure. *Journal of Financial Economics , 5* (2), 177-188.
Watsham, T. J., & Parramore, K. (1997). *Quantitative Methods in Finance.* London, Madrid etc.: International Thomson Business Press.
Verbeek, M. (2008). *A guide to modern econometrics.* Chichester: Wiley.
Wicksell, K. (1936). *Interest and prices.* London: Macmillan for the Royal Economic Society.
Wolfgang, L., & Theofanis, A. (2007). *Threshold dynamics of short-term interest rates: empirical evidence and implications for the term structure.* Frankfurt: Deutsche Bundesbank.
Woodford, M. (2003). *Interest and prices.* New Jersey: Princeton Univeristy Press.

9 APPENDIX

The set of equations that constituted the linear growth model was:

$$X_t = \mu_t + n_t; \quad \mu_t = \mu_{t-1} + \beta_{t-1} + w_{1,t}; \quad \beta_t = \beta_{t-1} + w_{2,t},$$

and the general form was:

$$X_t = \boldsymbol{h}_t^T \boldsymbol{\theta}_t + n_t; \quad \boldsymbol{\theta}_t = \boldsymbol{G}_t \boldsymbol{\theta}_{t-1} + \boldsymbol{w}_t.$$

Next, the computation phases from the linear growth model to the general form. Define $\boldsymbol{h}_t^T$ to be a row vector (1,0) and the matrix $\boldsymbol{G}_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and, as noted in the text, the state vector is given by $\boldsymbol{\theta}_t^T = (\mu_t, \beta_t)$, hence:

$$X_t = (0,1)(\mu_t, \beta_t) + n_t \Longrightarrow X_t = \mu_t + n_t$$

and the state vector is given by:

$$\boldsymbol{\theta}_t^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{t-1} \\ \beta_{t-1} \end{bmatrix} + \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix}$$

$$\Longrightarrow \mu_t = \mu_{t-1} + \beta_{t-1} + w_{1,t}; \quad \beta_t = \beta_{t-1} + w_{2,t}.$$